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MECHANICS

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PART I.

DYNAMICS AND STATICS.

BY THE

John Bascombe

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FELLOW AND BURSAR OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

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PREFACE.

THE following pages contain the more elementary parts of the Dynamics of a Particle and of the Statics of Parallel Forces, arranged with some additions from my Elementary Dynamics and Statics.

It is not possible to attain to a clear understanding of the elementary ideas of a difficult subject like Dynamics without patient thought, and without the careful working of numerical examples. Accordingly I have not thought it advisable to omit anything from the introductory chapters of the Dynamics except a few of the algebraical examples. I have given in Art. 16 a new form of that proof of the formula of accelerated motion which depends upon the idea of average velocity. The proof usually given is in my opinion unsatisfactory as taking too much for granted.

The Static Theory of Parallel Forces is deduced very simply from the Theory of Moments by the consideration of a system of parallel forces as a particular case of forces meeting at a point.

A combination of Morin's and Atwood's machines which I have suggested on p. 46 will I think be found useful by way of illustration. The chapter on the Pendulum is abbreviated from the latest edition (1891) of my Dynamics.

I may mention that in view of the publication of this book, the scope of my Dynamics has been somewhat enlarged in the latest edition by the addition of a chapter on Simple Harmonic Motion and the Pendulum, of some articles on Units and other minor points.

The Dynamical portion of the book is treated in a manner so different to that which has of late been customary in elementary text-books that I venture to ask Teachers and others who may use the book kindly to favor me with suggestions or criticisms. Of the value of this method of treating the subject as an introduction to the study of Mechanics I have received the very highest testimony from many Teachers—but there must be in a new book of this kind many details capable of improvement and many faults of omission besides other defects. Any communication on the subject will be gratefully acknowledged by myself or by the Publishers.

In arranging the book I have especially had in view the requirements of the Science and Art Department at South Kensington, but I hope that the book will be found a suitable introduction to the subject by other classes of Students as well.

J. B. LOCK.

August, 1891.

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SECTION I.

RECTILINEAR MOTION.

CHAPTER I.

VELOCITY, ACCELERATION.

1. The **distance** of a point P from a point A is represented in direction and magnitude by the finite straight line joining A to P.

The point P is said to be *in motion relative to*† the other point A, when the distance of P from A is changing.

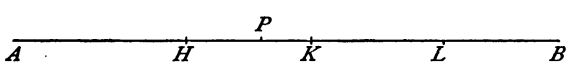
The distance of P from A may be changing in *length*, or in *direction*, or *both in length and in direction*.

While the *direction* of the distance of P from A remains unchanged the path of P is the straight line AP.

We shall in this section discuss the motion of a point which moves always in a straight line.

Accordingly, for the present, we are not concerned with any change of *direction* in the distances considered.

DEFINITION. The *change* in the position of the point P in any chosen interval of time is called the **displacement** of P for that interval.



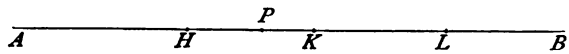
Draw a straight line AB; let this line extend any length beyond A and beyond B. Let a point P be in motion relative to A in the line AB. Then the distance AP is different at different instants. Thus, suppose the point P at one instant to be at H; at another, at K; at another,

† NOTE. All motion is *relative*, because there is no point in the universe which we have any reason for considering to be absolutely *at rest*.

at L; H, K and L being points *whose distances from* A *do not change*; then, P being in motion relative to A, P *alters its distance from* A *by the displacement* HK in the course of a certain interval of time.

This is what is meant when it is said that the displacement of P in a certain interval of time is HK.

We choose **a foot** as our *unit distance.*



NOTE. It must be carefully noticed that if in a certain interval the moving point goes from H to L and then *back* to K, *its displacement in that interval* is HK.

Thus $HL+LK=HK$; but $HL-KL=HK$.

Hence the displacement $LK=-$ the displacement KL.

Thus the displacement from L to K, in the *opposite direction*, is to be reckoned as of **opposite sign** to the displacement from H to L.

2. In what follows we shall frequently use the word **interval** as an abbreviation for *interval of time.*

An *interval* is a *duration* of time, which begins *at* a certain **instant** and ends *at* another *instant*; just as a *line* is *length* which begins at one *point* and ends at another.

We choose **a second** as our *unit interval.*

Let the displacements of the above moving point P be observed for a series of equal intervals of time. Then if its motion is of such a kind that in each of those equal intervals the point P has equal displacements, and if this is true whatever be the length of these equal intervals, then we say that the point P is moving **uniformly**; hence,

3. DEFINITION. The motion of a point P relative to another point A is said to be **uniform** when it is such that the point P has equal displacements in equal intervals of time, however short those intervals may be.

Suppose that the point P is moving uniformly in the straight line AP. Its **velocity** is *something* which is doubled, trebled, etc. if its displacement in a given interval is doubled, trebled, etc., and which is halved, divided by three, etc. if the interval required for a given displacement is doubled, trebled, etc.; in other words,

4. DEF. The **velocity** of a point which is moving uniformly in a straight line is proportional to the displacement of the point in a unit interval.

Velocity is time rate of displacement.

Velocity is therefore something which we observe in a moving point. It is capable of measurement; for we can say that the velocity of a point is double, or that it is half, or that it is some multiple, of the velocity of another point.

We therefore choose a certain velocity as our *unit* velocity, and give it a name; just as we choose a unit of length and call it a foot; and a unit interval and call it a second... ; hence

5. DEF. We choose as our **unit** *velocity* the velocity of a point which moving uniformly is displaced 1 foot in the interval 1 second.

We shall call this *unit velocity* a **velo.**

DEF. A **velo** in a given direction, is the velocity *a foot per second* in that direction.

The word *per* here denotes *for every.*

Example i. A point which moves at the rate of 3 feet per second has a velocity 3 times as great as that of a point which moves at the rate of 1 foot per second. Hence its velocity is 3 *velos.*

Example ii. *Express in* velos *the velocity of a point which moves uniformly at the rate of* 60 miles per hour.

In 1 hour the displacement is 60 miles;

that is, in 60×60 seconds the displacement is $60 \times 1760 \times 3$ feet; but it passes over an equal distance in each second;

therefore in 1 second the displacement is $\frac{60 \times 1760 \times 3}{60 \times 60}$ feet; that is, 88 feet.

Hence, its velocity is 88 velos.

Example iii. *A point has the uniform* velocity 8 *velos; how many miles does it pass over in an hour?*

The point has 8 velos; therefore, in 1 sec. its displacement is 8 feet; and it has equal displacements in equal intervals;

therefore in 60×60 seconds its displacement is $60 \times 60 \times 8$ feet;

that is, $\frac{8 \times 60 \times 60}{1760 \times 3}$ miles; or, $5\frac{5}{11}$ miles.

NOTE. It is most important to notice that a velocity requires an *interval of time* in which to produce a *finite* displacement however small.

A point P may have velocity *at* a certain instant, but unless the velocity continues during some interval after that instant, the point has no finite displacement from its position at that instant.

For example, a bullet which strikes a target is said to have a certain velocity **at** the instant at which it strikes.

EXAMPLES. I.

N.B. In each of the following questions the velocity given is uniform.

1. If I run 100 yards in 11 seconds, what is my velocity in velos?

2. A man runs at the rate of 7 velos; how long would he take to run a mile with this velocity?

3. Express in miles per hour (i) 40 velos, (ii) 100 yds. per minute.

4. How many velos has the extremity (i) of the minute hand of a clock which is 1 foot long, (ii) of the hour hand which is 10 in. long. [*The circumference of a circle* $= \frac{22}{7}$ *of the diameter.*]

5. How many velos has a man standing at the equator, in consequence of the daily revolution of the earth about its axis, the diameter of the earth being, say, 8000 miles?

6. The velocity of sound is 1118 velos; how long does it take to travel 13 miles?

7. Which is the greater velocity, 60 miles an hour or 500 yards in $11\frac{1}{4}$ seconds?

8. Find the ratio of the velocities $2\frac{1}{2}$ miles in 3 minutes and 10 ft. in a quarter of a second.

9. The following are 'records' in foot racing: (i) 100 yds. in 10 secs. (ii) a quarter of a mile in 49 secs. (iii) a mile in 4 min. $12\frac{3}{4}$ secs. (iv) ten miles in 51 min. $6\frac{1}{4}$ secs.; express the average velocity of each in velos.

6. A point which has v velos is displaced v feet in 1 second; it therefore is displaced (t times v) feet in t seconds.

Hence, if s be the number of feet in the displacement for t seconds of a point which has v velos, then

$$s = vt.$$

Example. A point passes uniformly over c miles *in* b hours; *express its velocity in* velos.

In $b \times 60 \times 60$ seconds its displacement is $c \times 1760 \times 3$ feet, therefore

in 1 second its displacement is $\frac{1760 \times 3 \times c}{60 \times 60 \times b}$ feet;

that is, it is displaced $\frac{22c}{15b}$ feet per sec.;

therefore it has $\frac{22c}{15b}$ velos.

7. The **measure** of the velocity of a point (when a velo is the unit) is the *number* of velos which it has.

This is, the *number* of feet in the displacement divided by the *number* of seconds in the interval, [provided the

velocity is *uniform*]. For, with the notation of Art. 6,

$$s = vt;$$

therefore

$$v = \frac{s}{t}.$$

This is what is meant when it is said that *The uniform velocity of a point is numerically equal to the displacement divided by the interval.*

EXAMPLES. II.

N.B. *The velocities given are uniform.*

1. Express in velos a velocity of c miles per hour.
2. A point has v velos; how far does it go in k hours?
3. A point has k velos; how long does it take to go m yards?
4. A point goes m yards in t seconds; how many miles does it go in k hours supposing its velocity uniform?
5. A point goes λ inches in 1 minute; how far does it go in k days at the same rate?
6. A point goes k feet in t seconds; how long does it take to go m miles with the same velocity?
7. A train goes n miles in k hours; how far does it go in t seconds?

AVERAGE VELOCITY.

8. DEF. The **average velocity** of a moving point *for a stated interval* (during which interval it is *not* moving *uniformly*), is the velocity of another point which, *moving uniformly*, has an equal total displacement in that interval.

Example. A point has the displacements 6 ft., 7 ft., 8 ft. and 9 ft. *respectively, in four consecutive* seconds; *find its* average velocity *for the four seconds.*

In 4 seconds its total displacement is $(6+7+8+9)$ ft.; that is, 30 ft.

Thus, in 4 seconds a point having the required average velocity passes over 30 feet;

therefore in 1 second this point passes over $7\frac{1}{2}$ feet;

that is, the required average velocity is $7\frac{1}{2}$ velos.

EXAMPLES. III.

1. A point has displacements of 9 ft., 10 ft., 11 ft. and 12 ft. in 4 consecutive seconds; find its average velocity for the 4 seconds.
2. Find the average velocity of the point in Question 1 (i) for the first 3 seconds, (ii) for the last 3 seconds.

3. A point has displacements of 20 yds., 24 yds., 28 yds., 32 yds. and 36 yds. in 5 consecutive seconds, shew that the average velocities for the 5 seconds, for the 3 middle seconds and for the 1 middle second are all equal.

4. A point has an average velocity of 2 velos, 3 velos, 4 velos, 5 velos, and 6 velos respectively in 5 consecutive intervals of 5 seconds each; find the average velocity for the 25 seconds; how far does the point go in the 25 secs.?

5. A point moves over a ft., $2a$ ft., $3a$ ft. and $4a$ ft. respectively in 4 consecutive intervals of b seconds each; find its average velocity for the whole time.

6. A point has an average velocity of $13v$ velos, $11v$ velos, $9v$ velos, $7v$ velos and $5v$ velos respectively in 5 consecutive intervals of t seconds each; find the average velocity for the $5t$ seconds.

7. A point has an average velocity of 3 velos for 2 secs., 6 velos for the next 4 secs., 10 velos for the next 4 secs. and 15 velos for the next 6 secs.; shew that in the 16 seconds its displacement is the same as that of a point which has an average velocity of 3, 5, 7, 9, 11, 13, 15, 17 velos respectively in 8 consecutive intervals of 2 seconds each.

8. A point is displaced 5 ft., 3 ft., 1 ft., -1 ft., -3 ft., in 5 consecutive seconds; shew that its average velocity is 1 velo for the 5 seconds.

VELOCITY WHICH IS NOT UNIFORM.

9. Consider a point in motion *not* moving *uniformly*.

Its average velocity during any interval is measured by its displacement in that interval divided by the length of the interval.

Its average velocity is different for different intervals.

It is the *average* velocity of a train for a certain interval, which is found by noting the number of seconds occupied by the train in passing successive quarter-mile posts; but it is not the average velocity which is meant when we speak of the velocity of a point **at** a certain instant.

10. The student must carefully distinguish between the velocity of a point **at** *a given* **instant** and its average velocity **for** *a given interval.*

Consider two *very long* trains moving on parallel lines in the same direction. Let one of the trains (P) be moving with a constant velocity, say, 20 velos; let the other train (Q) start from rest and move gradually faster and faster. A passenger in train P will see his train gain on train Q at first; [the train Q is supposed to be so long that during the time under consideration some part of it is alongside of the passenger in P]; but as time goes on, the train Q goes faster, until at last there comes an

instant at which the two trains are moving at exactly the same rate and neither is gaining on the other; this is only for an instant; it occurs *at* a certain instant. The train Q immediately begins to go faster than the train P; but there is a particular instant (the instant already referred to) *at* which neither train is going faster than the other and the two trains are *relatively at rest.* At that instant we say, that the velocity of the train Q is equal to the velocity of the train P; that is, the velocity of the train Q *at* that instant is 20 velos; hence,

DEF. The velocity of a moving point (which is not moving uniformly) **at a given instant,** is the velocity of another point which, moving with uniform velocity in the same direction, is at that instant *at rest relatively* to the first moving point.

UNIFORMLY INCREASING VELOCITY.

11. The simplest case of a point moving with velocity which is not uniform, is that of a point whose velocity *steadily* increases or whose velocity *steadily* decreases.

When a point is said to have velocity which is increasing or decreasing, the point does not move for an interval with uniform velocity, and then suddenly move with a different velocity; but the velocity *at* any instant gradually increases or decreases as time goes on.

12. DEF. A point moves with **uniformly increasing** velocity when its velocity is increased by equal increments in equal intervals of time, no matter how large or how small these equal intervals may be.

Uniformly increasing velocity is that which *grows* steadily; it does not increase by jerks or steps.

Suppose that a point P is moving with uniformly increasing velocity; suppose that at a certain instant its velocity is 100 velos, and that after an interval of 1 second its velocity is 101 velos; then at the end of one more second its velocity will be 102 velos; at the end of another half-second its velocity will be $102\frac{1}{2}$; and so on. Its velocity increases *at the rate* of 1 velo per second.

Example. A point moving with uniformly increasing velocity has velocities 20 velos *and* 28 velos *respectively at the beginning and at the end of an interval of* 12 seconds; *find its velocity* 3 seconds *before the beginning of the interval and* 3 seconds *after the end of the interval.*

In 12 seconds the velocity increases by 8 velos; therefore
in 1 second the velocity increases by $\frac{8}{12}$, or $\frac{2}{3}$ velo.

Hence, 3 seconds before the instant at which it has 20 velos it has $(20 - 3 \times \frac{2}{3})$ velos, or 18 velos. And 3 seconds after the instant at which it has 28 velos, it has $(28 + 3 \times \frac{2}{3})$ velos, or 30 velos.

13. A similar definition applies to the expression **uniformly decreasing** velocity.

Example. A point moving with uniformly decreasing *velocity has at a certain instant* 30 velos; *after* 5 seconds *it has* 20 velos; *when will it come to rest? when will it again have* 30 velos?

The velocity diminishes by 10 velos in 5 seconds,

Therefore it diminishes by 30 velos in 15 seconds.

When it has diminished by 30 velos it has 0 velo and is at rest. Therefore it comes to rest in 15 secs. from the instant at which it has 30 velos.

[When the velocity of a point is uniformly *decreasing*, we may consider the motion to be such that velocity in the **opposite direction** to its initial velocity is being continuously added. This continuous growth of *negative* velocity does not necessarily cease when the point comes to rest, Hence,] the velocity is still continuously *diminished* at the rate of 2 velos per second after the point has come to rest, that is, it then moves *backwards* from rest with velocity uniformly increasing at the rate of 2 velos per sec.; therefore after 15 seconds more it will be moving in the *opposite direction* with 30 velos.

EXAMPLES. IV.

N. B. *In the following examples the motion is in each case that of uniformly increasing or decreasing velocity.*

1. A point has 8 velos at the beginning and 9 velos at the end of a certain second; how many velos has it after 4 more seconds?

2. In a certain interval of half a minute the velocity of a point increases from 10 velos to 100 velos; what was its velocity at the middle of the interval?

3. In 1 hour the velocity of a point decreases from 300 velos to 120 velos; what was its velocity at the end of each quarter of that hour?

4. At noon a point is moving with 20 velos; at 4 P.M. it has 100 velos; what velocity has it at 2.15 P.M.?

5. At 2 o'clock a point has 7 velos; at 2.45 it has 142 velos; what has it at 3.30?

6. A train is being pulled up, and moves with uniformly decreasing velocity; at a certain instant it is going 60 miles per hour, after 18 seconds it has 80 velos; when will it stop?

7. A point which started from rest has after 4 seconds a velocity of 100 yds. per minute; when will it have 10 velos? and when will it be going at the rate of 60 miles an hour?

8. A point at a certain instant has 11 velos; 2 minutes later it is moving with a velocity of 30 miles an hour; when was it at rest?

9. At a certain instant a point has 128 velos and its velocity is decreasing at the rate of 32 velos per second; when will it come to rest? and how long will it be before it again has 128 velos?

10. A point moving at a certain instant at the rate of 11 ft. per second has its velocity increased by (7 ft. per second) per second; when will it be moving at the rate of 60 miles an hour?

11. A point which is moving at a certain instant with a velocity which is decreasing at the rate of (32 ft. per second) per second comes to rest after an interval of 3 seconds; what was its velocity at the beginning of the 3 seconds? and what at the middle of the 3 seconds? and what was it after 3 seconds more?

12. A point whose velocity is increasing at the rate of 32 ft. per second per second is moving at a certain instant vertically downwards with a velocity of 112 ft. per second; when was it at rest? and when was it moving vertically upwards with a velocity of 112 ft. per second?

14. When a point is moving with velocity which is uniformly increasing, its *average* velocity for a given interval of time is greater than the velocity which the point has at the *beginning* of the interval and is less than the velocity which the point has at the *end* of the interval.

For its velocity at the beginning of the interval is less than its velocity at any other instant in the interval; and its velocity at the end of the interval is greater than its velocity at any other instant in the interval.

Example i. *At a certain instant a point has* 8 velos *and its velocity is uniformly increasing* at the rate of 2 velos per second; *shew that in the next* four seconds *its displacement is greater than* 44 feet *and less than* 52 feet.

Its velocity at the *beginning* of each second is 8, 10, 12, 14 velos respectively; therefore its displacement is *greater* than $(8+10+12+14)$ feet in these 4 seconds.

Its velocity at the *end* of each second is 10, 12, 14, 16 velos respectively; therefore its displacement is *less* than $(10+12+14+16)$ feet in these 4 seconds.

Example ii. *Shew, by taking the velocity of the point in* Example i *at the beginning and at the end of each* quarter second, *that the total displacement in the* 4 seconds *lies between* 47 feet *and* 49 feet.

At the *beginning* of each $\frac{1}{4}$ second the velocities are

$8, 8\frac{1}{2}, 9, 9\frac{1}{2}, 10, 10\frac{1}{2}$ etc....up to $15\frac{1}{2}$, velos, respectively,

and the displacement is therefore *greater* than

$$\tfrac{1}{4}\{8+8\tfrac{1}{2}+9+9\tfrac{1}{2}+\ldots\ldots\ldots\ldots+15\tfrac{1}{2}\}\text{feet},$$

that is, *greater* than (the sum of the series in A.P., viz.) 47 feet.

At the *end* of each $\frac{1}{4}$ second the velocities are

$8\frac{1}{2}, 9, 9\frac{1}{2}$ etc....up to 16, velos, respectively,

and the displacement is therefore *less* than

$$\tfrac{1}{4}\{8\tfrac{1}{2}+9+9\tfrac{1}{2}+\ldots\ldots\ldots\ldots+16\}\text{ feet},$$

that is, *less* than 49 feet. Q. E. D.

EXAMPLES. V.

N. B. *The following velocities are each uniformly increasing or decreasing.*

1. At a certain instant a point has 8 velos and its velocity is increasing uniformly at the rate of 1 velo per second; shew by considering the velocity at the beginning and at the end

(i) of each second, that its total displacement in the next 10 seconds is greater than 125 ft. and less than 135 ft.

(ii) of each $\frac{1}{10}$th of a second, that its displacement in the 10 seconds is greater than $129\frac{1}{2}$ ft. and less than $130\frac{1}{2}$ ft.

2. At a certain instant a point has 120 velos and its velocity is decreasing at the rate of 32 velos per second; shew by considering the velocity at the end of each $\frac{1}{32}$th of a second that its displacement in the next second is less than $104\frac{1}{2}$ ft. and greater than $103\frac{1}{2}$ ft.

3. At a certain instant a point has 16 velos and its velocity is increasing at the rate of 32 velos per second; shew by dividing the interval of time into $\frac{1}{32}$ths of a second that its displacement in the next 3 seconds cannot differ from 192 feet by more than 18 inches.

4. At a certain instant a point is at rest and velocity is given to it uniformly so that at the end of 1 second it has 32 velos; shew as in Example 3, that in the 3rd second of its motion its displacement does not differ from 80 feet by more than 6 inches.

15. It is of the greatest importance that the student should mentally realize the motion of a point moving with uniformly increasing (or decreasing) velocity.

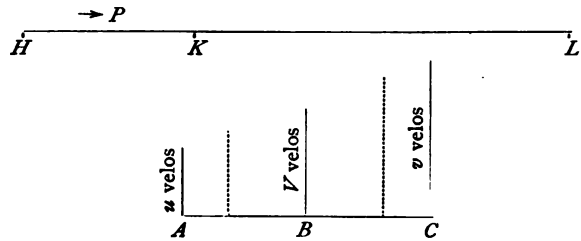
Consider again the case of two very long trains A and B moving on parallel lines in the same direction; suppose them to be passing a certain station H,—the train A with a constant velocity 20 velos,—the train B with a constantly increasing velocity which, when the engine is at H, is 4 velos.

At H suppose the engines are abreast; at first A will gain upon B; but it will gradually gain more and more slowly, until an instant arrives when the two trains are relatively at rest; after which time B will at once begin to gain upon A; and will gradually gain at a faster and faster rate.

Hence an instant arrives when the engines are again abreast; suppose this to occur at a place K; then each train has passed over the distance from H to K in the same interval; hence the average velocity of B *for the time occupied in going from* H *to* K is the velocity of A, viz. 20 velos.

It will be found that *the instant at which* these two trains are relatively at rest, *is at the middle of the interval of* time which they each take in going from H to K. (So that the engine of the train A, which moves with uniform velocity, is *midway* between the stations, while the engine of the train B has not yet reached the middle point.) That is

16. PROP. *When the velocity of a point* P *is uniformly increasing, its* **average velocity** *for any given interval of time is equal to the velocity of* P **at the middle instant of that interval of time.**



Let *HKL* represent the path of the moving point *P*.

Let the line *ABC* represent the interval of time; let *B* be the middle instant of the interval.

Let the velocity of *P* at the instants *A*, *B*, *C* respectively be *u* velos, *V* velos, *v* velos. Let the position of *P* at the instants *A*, *B*, *C* respectively be *H*, *K*, *L*.

At the instant *B*, (when *P* arrives at *K* with *V* velos) let a point *Q* be starting from *K* towards *H* with *V* velos, and let the subsequent velocity of *Q* towards *H* be such that it continues to separate from *P*, as *P* moves towards *L*, with the *uniform* velocity 2 *V* velos.

Then since *P* moves with uniformly *increasing* velocity, *Q*, moving in the opposite direction, must have velocity which uniformly *decreases at the same rate.* Hence *Q* moves exactly as if *P*, on its arrival at *K* suddenly turned back and retraced its path with velocity exactly reversed at each point of its path. Therefore at the end of an interval equal to *AB*, *Q* will arrive at *H*.

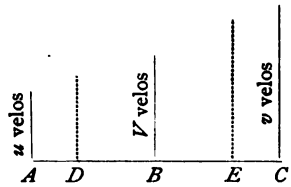
But, since $BC = AB$, *P* arrives at *L* at that same instant.

Now *P* and *Q* *separate* from each other with uniform velocity 2 *V* velos; therefore their total separation, *HK*, in the interval *BC* is the displacement due to the uniform velocity 2 *V* velos.

Therefore *HK* is the displacement due to the uniform velocity *V* velos for the interval *AC*. Q. E. D.

17. *The above proof may be summed up as follows.*

Let ABC represent the interval of time; let B be the middle instant of the interval.



Consider the velocities which the point P has at two instants D and E equidistant from the middle instant B.

The velocity of P at D is just as much *less* than the velocity V velos of P at B, as the velocity of P at E is *greater* than V velos.

So that the sum of the velocities at those two instants is $2V$ velos; this is true for every such pair of instants; and since the two intervals AB, BC from which the pairs of instants are taken one from each are equal, it follows that the average velocity of P for the whole interval AC is V velos. Q. E. D.

Example i. *At a certain instant a point has* 18 velos, *and it is moving with uniformly increasing velocity, so that* 1 second *later it has* 20 velos; *how far does it go in that second?*

The average velocity for the 1 sec. is the same as the velocity which the point has at the middle of the interval, that is, after half a second.

Since in 1 second the velocity is increased by 2 velos,

therefore in $\frac{1}{2}$ second the velocity is increased by 1 velo;

hence the average velocity for the 1 second is $(18+1)$, or, 19 velos.

In 1 second it passes over 19×1 ft., or 19 feet.

Example ii. *How far does the point in* Example i. *go in* 8 seconds?

The middle of the interval of 8 seconds is 4 seconds after the time when it has 18 velos. In 1 second the velocity is increased by 2 velos, therefore in 4 seconds the velocity is increased by 8 velos;

hence, the average velocity for the 8 seconds is $(18+8)$ or 26 velos;

therefore, its displacement is 26×8 ft. that is, 208 feet.

Example iii. *A point, moving with uniformly decreasing* velocity, *is displaced* 294 ft. *in a certain interval of* 7 secs.; *at the beginning of the* 7 seconds *it has* 49 velos; *find its* velocity *after* 3 seconds *more.*

Since it passes over 294 ft. in the 7 seconds, its average velocity [Art. 8] is 42 velos; its actual velocity at the middle of the 7 secs. is equal to this average velocity; hence, since at the beginning of the 7 seconds it has 49 velos, and $3\frac{1}{2}$ seconds later it has 42 velos, therefore its velocity decreases by 7 velos in $3\frac{1}{2}$ secs.; or, by 2 velos per second;

hence, after 10 seconds it will have (49 – 20) or 29 velos.

EXAMPLES. VI.

N. B. *In the following examples the velocity is uniformly increasing (or decreasing).*

1. A point has 100 velos at 12 o'clock, and its velocity increases to 104 velos in 1 second; find

(i) how far it goes in 1 second, (ii) how far it goes in 2 seconds,

(iii) how far it goes in the first 30 secs. after 1 min. past 12 o'clock,

(iv) how far it goes between 12.1′ and 12.2′ o'clock.

2. At a certain instant a point has 16 velos, and its velocity increases at the rate of 32 velos per second; shew that in the next three seconds its displacement is 192 ft. [See Ex. V. 3.]

3. At a certain instant a point has 120 velos, and its velocity is decreasing at the rate of 32 velos per second; shew that in the next second its displacement is 104 ft. [See Ex. V. 2.]

4. A point in a certain interval of 3 seconds passes over 300 feet; in the next 4 seconds its displacement is 764 ft.; shew that its velocity is increasing at the rate of 26 velos per second.

5. A point in a certain interval of 5 seconds passes over 125 feet; in the next 8 secs. it passes over 304 ft.; when did it start from rest?

6. A point is observed to pass over 120 ft., 129 ft., 138 ft. respectively in 3 consecutive seconds; is this consistent with uniformly increasing velocity?

7. Starting from rest, in the 5th second of its motion, a point passes over 144 ft.; find its velocity after 10 seconds from rest.

8. A point moves 144 ft. in a certain interval of 2 secs. and comes to rest after 3 secs. more; how much further did it move?

9. A point starts from rest, and after 10 seconds it has 25 velos; how far does it go in the 20th second of its motion?

10. A point passes over 24 ft. in a certain second; at the end of that second it has 28 velos; how far does it go in the next second? and when is it at rest?

UNIFORM ACCELERATION.

18. DEF. When a point is moving with uniformly increasing velocity, its **acceleration** is *that* which is proportional to the *increase* of velocity in a unit interval.

Thus, the acceleration of a point is doubled, or trebled etc. when the velocity added in a given interval is doubled, or trebled etc.; and it is halved, or divided by three, etc. when the interval (in which a given velocity is added) is doubled, or trebled etc.

We choose as our **unit** *acceleration*, the acceleration of a point which, moving with uniformly increasing velocity, has its velocity increased by 1 velo in the course of each second.

We shall call this *unit acceleration* a **celo**.

DEF. A **celo** is (*a foot per second*) *per second.*

19. In the present work we consider only **uniform** acceleration. So that when a point is said to have a certain number of celos, it is implied that the point has that uniform acceleration *throughout* the interval under consideration.

NOTE. It is most important to notice that a point which has acceleration requires **an interval of time** in which to increase its velocity by any definite amount.

Example i. A point which has 6 velos added to its velocity in the course of each second, has six times the acceleration of a point having the unit acceleration; hence it has 6 *celos*. The same point has 1 velo added in the course of the $\frac{1}{6}$th part of a second.

Example ii. *A point which has* 6 celos *has at a certain instant* 12 velos; *how many* velos *will it have* 10 seconds *later?*

In the course of 1 second 6 velos are added,

therefore in 10 seconds 60 velos are added.

So that after 10 secs. the point has $(12+60)$ velos, or 72 velos.

Example iii. *How far does the point in* Ex. ii. *go in those* 10 secs.?

The average velocity of the point for the 10 seconds is that which it has at the middle of the interval; that is, $(12+5\times 6)$ velos, or 42 velos; therefore, in 10 seconds the point goes 42×10 ft., or 420 feet.

Example iv. *A point which has* 32 celos *starts* from rest; *how far will it go in the* 4th second *of its motion? and how far in the* 12th second?

At the middle of the 4th second, that is, after $3\frac{1}{2}$ secs., it has $3\frac{1}{2} \times 32$ velos, or 112 velos; hence,
for the 4th second its average velocity is 112 velos;
therefore, in that second its displacement is 112×1 ft., or 112 ft.

Again, at the middle of the 12th second; that is, at the end of $11\frac{1}{2}$ secs. it has $11\frac{1}{2} \times 32$ velos, or 368 velos; therefore,
for the 12th second its average velocity is 368 velos;
so that in that second it goes 368 ft.

Example v. *A point which at a certain instant has u* velos, *has u'* velos *added on in the next t* seconds; *what is its* acceleration?

When 1 velo is added in 1 second the point has 1 celo;

therefore, when 1 velo is added in t seconds the point has $\frac{1}{t}$ celos,

and when u' velos are added in t seconds the point has $\frac{u'}{t}$ celos.

Therefore the required acceleration is $\frac{u'}{t}$ celos. Hence

20. The *measure* of the acceleration of a point is the **number** of units of velocity added on per unit-interval.

21. When the velocity of a point is *decreasing* uniformly it is said to have **negative acceleration.**

When the acceleration of a point is opposite in direction to its velocity it is often called **retardation.**

Suppose a point which has a uniform acceleration to have at a certain instant 32 velos and at the end of 1 second to have 30 velos, we say that it has -2 celos. This is consistent with the actual result and is a very convenient way of speaking. [See Art. 13.]

This point at the end of 5 seconds will have $(32 - 2 \times 5)$ velos.

EXAMPLES. VII.

N.B. *In these examples the motion in each case is uniformly accelerated in one straight line; so that when a point is said to have a certain number of celos, it retains that acceleration* throughout *the motion under consideration.*

1. A point which has 32 celos, has at a certain instant 16 velos, find (i) its velocity after 2 seconds, (ii) its velocity after 5 seconds,
(iii) how far it goes in the first 4 seconds,
(iv) how far it goes in the first 20 seconds,
(v) when it will have 1100 velos, (vi) when it was at rest.

2. At a certain instant a point has 200 velos; 5 seconds later it has 300 velos; find its acceleration.

3. A point which has 32 celos is displaced 144 ft. in a certain interval of 3 seconds; shew that it started from rest.

4. A point which has 32 celos passes over 144 ft. in a certain interval of 3 seconds; how far will it go in the next 4 seconds?

5. In a certain interval of 5 seconds a point goes 250 ft. and in the next 8 seconds it goes 608 ft. further; find its acceleration.

6. A point which has a certain acceleration is at a certain instant at rest; during the tenth second from rest it passes over 304 ft.; find
(i) its acceleration,
(ii) its displacement in the 5th second reckoning from the instant at which it was at rest.

7. A point which has -5 celos, has at a certain instant 30 velos; find (i) its velocity at the end of 3 seconds,
(ii) when and where it comes to rest,
(iii) its displacement in the first 5 seconds,
(iv) the ratio of the distance passed over in the first 3 seconds to that passed over in the next 3 seconds.

8. In a certain interval of 3 secs. a point is displaced 288 ft.; in next 2 secs. it is displaced 32 ft.; find
(i) its acceleration, (ii) when it is at rest,
(iii) where it is after 3 more seconds,
(iv) when it has 144 velos, (v) when it has -144 velos.

9. A point is displaced h feet in t seconds; in the next $2t$ seconds it is displaced k feet; find its acceleration.

10. A point passes over h feet in 2 seconds; after an interval of t seconds it is observed to be displaced k feet in 2 seconds; find its acceleration.

*22. PROP. *Let a point have* u velos *at the beginning of a certain interval of* t seconds; *let it have* a celos *throughout the interval; and let* s feet *be its displacement in the* t seconds; *then* $s = ut + \frac{1}{2}at^2$.

For, by Art. 17, the *average velocity* for the t seconds is the velocity which it has at the middle of the interval; this is the velocity at the end of $\frac{1}{2}t$ seconds, and is

$$(u + \tfrac{1}{2}at) \text{ velos.}$$

Therefore the displacement for the t seconds is

$$[(u + \tfrac{1}{2}at) \times t] \text{ feet.}$$

Hence $s = ut + \frac{1}{2}at^2$. Q. E. D.

*23. The results of Arts. 17 and 22 may also be proved by the method of Examples V. 6 as follows.

PROP. *To prove, with the usual notation, that*

$$s = ut + \tfrac{1}{2}at^2.$$

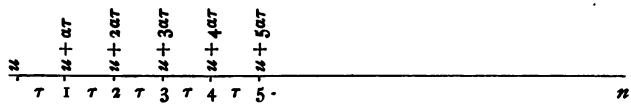
Let the interval t seconds be divided into n equal intervals each of τ seconds; so that, $n\tau = t$.

The velocities at the beginning of each of these n intervals are respectively

u velos,
$(u + a\tau)$ velos,
$(u + 2a\tau)$ velos,
...
$(u + n - 1\ a\tau)$ velos.

The velocities at the end of each of these n intervals are respectively

$(u + a\tau)$ velos,
$(u + 2a\tau)$ velos,
$(u + 3a\tau)$ velos,
...
$(u + na\tau)$ velos.



Let s' ft. be the total displacement which the moving point *would* have supposing that during each interval it moved uniformly with its velocity at the *beginning* of each interval, and s'' ft. the total displacement which the moving point would have supposing that it moved uniformly in each interval with its velocity at the *end* of that interval.

Then, by Art. 14, s lies between s' and s''. But

$$\begin{aligned} s' &= u\tau + (u + a\tau)\tau + (u + 2a\tau)\tau + \ldots + (u + \overline{n-1}\ a\tau)\tau \\ &= nu\tau + a\tau^2[1 + 2 + 3 + \ldots + (n-1)] \\ &= un\tau + a\tau^2 \times \tfrac{1}{2}(n-1)(n) = ut + \tfrac{1}{2}an^2\tau^2 - \tfrac{1}{2}an\tau \times \tau \\ &= ut + \tfrac{1}{2}at^2 - \tfrac{1}{2}at\tau \qquad [\text{since } an\tau = t] \end{aligned}$$

$$\begin{aligned} s'' &= (u + a\tau)\tau + (u + 2a\tau)\tau + \ldots + (u + na\tau)\tau \\ &= nu\tau + a^2\tau^2(1 + 2 + 3 \ldots + n) = un\tau + a\tau^2 \times \tfrac{1}{2}(n)(n+1) \\ &= ut + \tfrac{1}{2}at^2 + \tfrac{1}{2}at\tau. \end{aligned}$$

This is true however small τ may be.

Let n be increased, and therefore τ diminished without limit; then s' and s'' both become $ut + \frac{1}{2}at^2$; and s is between s' and s''. Therefore $s = ut + \frac{1}{2}at^2$. Q. E. D.

24. *To prove that, if v be the* number of velos *which the moving point has at the end of the interval* t seconds, *then* $2\alpha s = v^2 - u^2$.

We have $v = u + \alpha t$ and $s = ut + \frac{1}{2}\alpha t^2$;

$$\therefore \quad 2\alpha s = 2u\alpha t + \alpha^2 t^2$$
$$= u^2 + 2u\alpha t + \alpha^2 t^2 - u^2$$
$$= (u + \alpha t)^2 - u^2 = v^2 - u^2. \qquad \text{Q. E. D.}$$

25. These three results are of great importance; viz.

$$\mathbf{v = u + \alpha t} \qquad \text{(i)},$$

$$\mathbf{s = ut + \tfrac{1}{2}\alpha t^2 = \frac{u + v}{2} \times t} \qquad \text{(ii)},$$

$$\mathbf{\alpha s = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2} \qquad \text{(iii)},$$

where **t** is the *number* of seconds in the interval under consideration,

u is the *number* of velos which the moving point has at the beginning of the *t* seconds,

v is the *number* of velos which the point has at the end of the *t* seconds,

α is the *number* of celos which the point has throughout the *t* seconds,

s is the *number* of feet in the displacement† of the point for the *t* seconds.

N. B. In future, when using the letters *s*, *u*, *a* or *t* we shall always suppose them to have the meaning given in Art. 22.

† NOTE. The number of feet given by the formula $s = ut + \frac{1}{2}at^2$ when *a* and *u* are of opposite sign is not always the distance *passed over* by the point, in the sense of *the number of feet travelled*. It is the *resultant* distance, or *displacement*, of the point from the initial point. In fact *s* gives the *position* of the point at the *end* of the interval *t* seconds.

Thus in Example ii. on page 21, $s = 1600$ when $t = 5$ and also when $t = 20$, but during the interval 20 seconds the point travels many more than 1600 feet; it travels 1600 feet in 5 seconds, then goes 900 feet further in $7\frac{1}{2}$ secs. and in the next $7\frac{1}{2}$ secs. comes back 900 feet; so that at the end of 20 secs. it is again at a distance 1600 feet from the initial point.

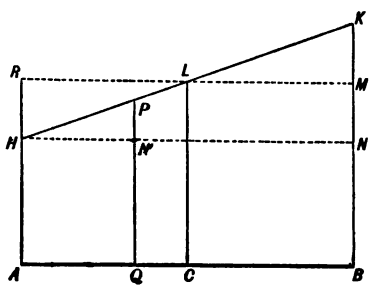
26. The following diagram which may be called a **Diagram of Velocity**, will be found to be useful in solving questions on motion.

We draw a horizontal line AB so that its length records the interval t seconds; this horizontal line may be looked upon as a part of the rim of a clock, straightened.

Different *points* on this line indicate different *instants*.

We record the *number* of velos which a point has at a particular instant Q by the *length* of a line QP drawn, from the point indicating that instant, at right angles to the horizontal line.

It must be most carefully noticed that there is no *line* in the diagram which indicates the *displacement*.



Let the diagram be that of a point moving with acceleration a and with initial velocity u.

Let $AH=u$, $BK=v$; join HK; bisect AB in C; draw CL parallel to AH; draw $HN'N$, RLM parallel to AB.

Then $$KN=v-u=at=a\times AB.$$

$$\therefore\ a=\text{the ratio of } KN \text{ to } AB \quad [=\tan KHN].$$

If the velocity at Q is represented by QP, then P must be in the line HK; for the additional velocity at Q which is $N'P=a.HN'$, and

$$\therefore\ NP:HN=a=KN:HN.$$

The displacement $s=t\times(u+\frac{1}{2}at)=AB\times(BN+NM)$

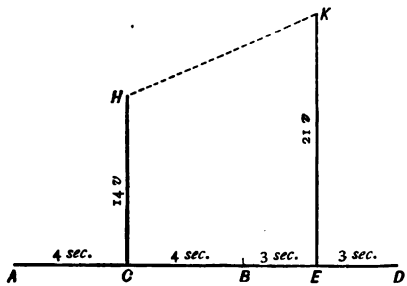
$=$the area of $RABM=$the area of $AHKB$

$=AB\times CL.$

$\therefore$ CL represents the *average velocity*.

Example i. *A point is displaced* 112 feet *in* 8 seconds *and it is displaced the next* 126 feet *in* 6 seconds; *find its* acceleration; *find also* how long *it will take to go the next* 162 feet.

Draw the velocity diagram; let AB represent 8 secs.; let BD represent 6 seconds.



The velocity *at* C, the middle instant of AB, is $\frac{112}{8}$ velos $= 14$ velos (Art. 17); represent this by CH.

The velocity at E, the middle instant of BD, is $\frac{126}{6}$ velos $= 21$ velos; represent this by EK. Join HK.

The velocity is uniformly increasing. It increases by $(EK - CH)$, or by 7 velos in CE, that is, in $(4+3)$ seconds, or, in 7 secs.

Therefore the point has 1 celo.

At the instant D the velocity is $(21+3)$ or 24 velos.

Hence, if the point is displaced the next 162 ft. in x seconds,

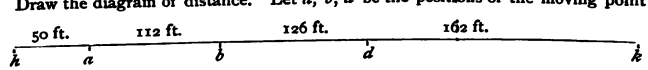
$$162 = 24x + \tfrac{1}{2}x^2; \text{ or, } x^2 + 48x = 324.$$

Whence $$x = 6 \text{ or } -54\dagger\ldots$$

That is the point takes 6 seconds to go the next 252 ft.

† This result indicates that an instant 54 seconds before the instant D (from which instant the interval x seconds is measured) the distance of the moving point (P) was 162 feet from its position at the instant D. Here we assume that the moving point P has 1 celo for an unlimited length of time.

Draw the diagram of distance. Let a, b, d be the positions of the moving point



at the instants A, B, D. Then $ab = 112$ ft., $bd = 126$ ft.; let $dk = 162$ ft.; let $ka = 50$ ft. Then it will be found that 54 seconds before D the moving point was at k, moving in the direction kd with 30 velos; 30 seconds later it came to rest at k; 10 seconds later it was at a; 14 seconds later it was at d; 6 seconds later it was again at k. During all this time it had 1 celo in the direction of kk.

Example ii. *A point has* -32 celos *and when it is in a certain position, it has* 400 velos, *find when it will be at the distance of* 1600 feet *from that point.*

Let the required interval be x seconds; then since $s=ut+\frac{1}{2}at^2$,

therefore, $1600=400x-16x^2$, or, $x^2-25x+100=0$,

or, $(x-20)(x-5)=0$. So that, either $x=20$, or $x=5$.

Thus the required interval is either 5 *seconds*, or 20 *seconds*.

The point has -32 celos; that is, its velocity is decreasing. In 5 seconds its displacement is 1600 ft.; then it goes on, comes to rest, and returns to the same point, after 15 seconds more.

Example iii. *A point which has* 5 celos *has initially* 20 velos; how far *will it go in attaining a velocity of* 100 velos?

Let the required distance be x feet; then since

$$as=\tfrac{1}{2}v^2-\tfrac{1}{2}u^2,$$

therefore, $5x=\frac{1}{2}(100^2-20^2)$,

or, $5x=5000-200$;

that is, $x=960$.

The required distance is 960 *feet*.

Example iv. *A point which has an* acceleration *of* 1 mile per minute per minute *is moving* at a certain instant *with a* velocity *of* 15 miles per hour; *how far does it go in the next minute?*

15 miles per hour $=22$ velos. [Ex. ii. p. 3.]

By an acceleration of 1 mile per minute per minute, is meant that an additional velocity of one mile per minute is produced in each minute.

Now, a mile per minute $=\frac{1760}{60}\times 3$ ft. per second,

that is, 88 velos.

Hence, 88 velos are added on in the course of each minute;

that is, $\frac{88}{60}$ velos are added on per second.

Hence, 1 mile per min. per min. $=\frac{88}{60}$ celos.

Therefore the point will move over

$$\{22\times 60+\tfrac{1}{2}\cdot\tfrac{88}{60}\times(60)^2\} \text{ feet in 60 secs.;}$$

that is, $(1320+2640)$ ft., or, 3960 ft.

It should be noticed that the complete motion of a point which moves *continuously* with constant acceleration, and which at a certain instant is at rest at a certain point, must consist before that instant of an approach to that point along a straight line and after that instant of a repassing back again over the same straight line with its velocity at each point equal and opposite to the velocity of its approach.

EXAMPLES. VIII.

The motion in each case is that of uniform acceleration in a straight line.

1. A point has -32 celos and at a certain instant it has 96 velos; in how many seconds will it be at a distance 144 ft. from the position which it has at that instant?

2. A point has 2 celos and initially it has 20 velos; how long will it take to go 100 ft.? and how far will it go before it attains the velocity 25 velos?

3. One point P has 32 celos and it has initially 200 velos; a second point Q has 16 celos and, initially, 1000 velos; initially they start from the same point in the same direction; when are they again together? and how far will they have gone?

4. A point which has 32 celos starts from rest; from the same place, at the same time and in the same direction another point which has 30 celos starts with an initial velocity of 40 velos; when and where will the first point overtake the second? and when will they have equal velocities?

5. A point having initially 20 velos attains a velocity of 30 miles per hour in passing over 128 ft.; find its acceleration.

6. A point after going a quarter of a mile with an acceleration of 20 ft. per second per second, attains a velocity of 2704 ft. per second; what was its initial velocity?

7. A train in going a quarter of a mile increases its velocity from 5 miles an hour to 35 miles an hour; what is its acceleration?

8. A train moving at the rate of 45 miles an hour is stopped by the continuous brake in 220 yds.; what is its 'acceleration'?

9. A train has at a certain instant a velocity of 60 miles per hour and it has a retardation of one mile per minute per minute; how far will it go before it comes to rest?

10. A point in going a mile has its velocity increased from a mile in 3 minutes to a mile in $2\frac{1}{2}$ minutes: how many celos has it?

11. A point is displaced 100 yards in 10 seconds, and the next 100 yards in 12 seconds; when will it be 100 yards further on?

12. A point is displaced 200 ft. in 12 seconds, and the next 300 ft. in 20 seconds; when will it be 400 ft. further on?

13. A stone which has -32 celos is passing a certain point; at the point 48 feet further on it has 32 velos; in how many seconds will it return to the first point?

14. A point is observed to move a distance of 48 ft. from rest in 60 seconds; what is its acceleration?

15. A point has 32 celos; it has at a certain point a certain velocity in the opposite direction to its acceleration; it returns to that point after an interval of 4 seconds; find how far it travels from that point and its velocity at that point.

16. A point having a certain constant acceleration moves in a certain second over a distance of 20 ft. and in the next second but one it moves over 128 ft.; what is its acceleration?

17. A point having a constant acceleration is displaced 72 ft. while its velocity increases from 16 to 20 velos; what is its acceleration?

18. A point having a constant acceleration passes over in the fifth and the seventh second of its motion distances of 108 ft. and 140 ft. respectively; find its acceleration and its initial velocity.

19. A point having a certain constant acceleration attains a velocity of 40 miles an hour from rest in going 1 mile; the acceleration is then changed into a constant retardation so that the point comes to rest in 440 yards more; compare the magnitudes of the acceleration and retardation.

20. A point has 32 celos for 4 seconds; the acceleration is then reversed so as to be -32 celos; find how far the mass goes in 8 seconds from rest.

CHAPTER II.

MASS AND FORCE.

27. Consider some definite quantity of matter; say, a cubical lump of iron. This (i) consists of a certain *kind of material* (iron), (ii) has a definite *shape*, (iii) has a definite *volume*, (iv) contains a certain *quantity of matter*, (v) has a definite *weight*.

Each of these (i), (ii), (iii), (iv), (v) is a separate and distinct idea applicable to the lump of iron.

In dynamics we are chiefly concerned with the last two; viz., the *quantity of matter*, which we shall call the **mass**, and the *weight*.

We proceed to consider what is meant by *mass*.

28. We derive our idea of matter or mass from our muscular sense.

We also derive our idea of **force** from our muscular sense. When we exert our muscles, we say we are applying *force* to something; and that to which we apply force we call *mass*.

In order to make an experiment with force we must have some means of applying a constant force to a given mass.

We know that when we compress a coiled steel spring in a certain way we have to exert a definite amount of force; also it is known that a spring can be made which will resist the same amount of force for a very considerable time.

The great difficulty in making an experiment which shall test the effect of a single force when applied to a certain mass, is the difficulty of getting rid of *all other* forces (such as friction, weight, etc.) so as to leave only the single force to be considered.

Let us imagine a perfectly smooth, perfectly horizontal sheet of ice*. Upon this place a lump of matter, such as a smooth block of stone in the form of a cube. Now take a steel spring arranged in some manner so that its state of compression can be easily observed (a spring letter-weighing machine for instance), and fastening this to the centre of one of the horizontal faces of the lump, apply a constant horizontal force by means of it to the lump, in the direction perpendicular to its face, taking care to keep the force constant and to apply it continuously and uniformly to the lump for an interval of, say, 10 seconds.

What happens? It will be found that by the force the lump has a certain amount of **acceleration** given to it; so that a velocity grows in the lump *as long as the force is applied;* and thus at the end of the 10 seconds the lump will have acquired a certain number of velos.

At the end of the 10 seconds let the force be withdrawn.

What happens? The lump has a certain number of velos; and it will be found that it will continue to move uniformly with that number of velos, so long as there is nothing in the nature of a force applied to it.

29. Suppose now that we try the experiment of Art. 28 with *several* lumps of the same size but of different material; say, one of lead, one of stone, one of cork; and suppose that by the aid of three exactly similar steel springs we can apply an equal constant horizontal force to each lump continuously for an interval of, say, 10 seconds.

What happens? It will be found that the lead, the stone, and the cork, have each a constant number of celos communicated by the force; but that the number of celos given to the cork is greater than the number given to the stone, and the number of celos given to the stone greater than the number given to the lead. So that by the end of the 10 seconds they will each have acquired a certain number of velos, the cork more than the stone, the stone more than the lead†; also, that after the force has been withdrawn, each will continue to move with its own constant velocity, so long as there is nothing in the nature of a force applied to it.

* It must be noticed that these conditions can only be approximately fulfilled; the best ice is not perfectly smooth and the movement of the air would interfere to a certain degree with such an experiment.

† It is instructive to imagine the three blocks of the same size and shape and painted the same colour; the application of an equal horizontal force to each will at once reveal which is lead, which stone and which cork.

30. DEF. We choose as our **unit mass** the mass of a certain lump of metal [called 1 lb. (avoirdupois)].

We call this *unit mass* a **pound**, or, 1 **lb.**

31. DEF. **Force*** is that which when applied to *mass* produces in it acceleration in the direction of the force; so that the force varies as the acceleration which it produces in a given mass; and also varies as the mass in which it can produce a given acceleration.

Consider (i) the experiment of Art. 28; here, if we double the force, we shall find that we produce in the same mass double the acceleration; consider (ii) the experiment of Art. 29; here, if the stone has 5 times the '*mass*' of the cork, we must have a force 5 times as large acting on the stone as on the cork to produce in each the same number of celos.

32. DEF. We choose as our **unit force** that force which acting on *a pound* produces in it 1 celo.

We call this *unit force* a **poundal.**

A poundal produces one celo in one lb.;

or

Whenever a pound has one celo it is then under the action of one poundal.

33. The statement of Art. 31 is to be given the fullest possible interpretation; it asserts that

I. *Force* is *that* which produces *acceleration* in *mass*; therefore, whenever a mass has acceleration, it is then under the action of some external force.

II. When *no external force* acts on a mass it has *no acceleration*; in other words, if at rest, the mass remains at rest, and if in motion, the mass continues to move with uniform velocity.

III. When the force applied to a certain mass is doubled or trebled, etc., then the acceleration produced is doubled or trebled, etc.

* It is important to notice that the word **force** is in ordinary language used in more senses than one. The above definition adopts the word (*force*) as a *scientific* term and therefore limits its use to one definite idea only.

IV. When the mass to which a certain force is applied is doubled, or trebled, etc., then the acceleration is halved, or divided by three, etc.

V. Every force applied to a mass produces in that mass its proper acceleration in its own direction, independently of any other motion which the mass may have.

Thus, by III, Forces are measured by the number of celos which they respectively produce in 1 lb.

By IV, Masses are measured by the number of poundals required to produce 1 celo in each of them respectively.

And by V, when two equal and opposite external forces act on a particle, they produce equal and opposite accelerations; in which case the particle continues at rest, or moves with constant velocity.

34. A force which is *applied* to a mass A by another mass B is said to be **external** to A.

By definition, 1 poundal acting on 1 lb. produces in it 1 celo.

Therefore 1 poundal acting *continuously* for 1 second on 1 lb. adds 1 velo to its velocity.

By definition, 2 poundals acting on 1 lb. produce in it 2 celos.

and 2 poundals acting on 2 lbs. produce in it 1 celo.

Example i. *A force of* 3 poundals *is applied continuously to the mass* 1 lb. *which at the beginning of a certain interval has* 10 velos, *the force having the* same direction *as the velos; find* (i) *when the mass will have* 25 velos; (ii) how far *it will go in* 12 seconds.

By Art. 32, 3 poundals acting on 1 lb. produce in it 3 celos,

3 celos produce 15 velos in 5 seconds. (i)

The 1 lb. has 3 celos; the velocity of the 1 lb. at the middle of the 12 seconds is therefore $(10+3\times 6)$ velos; that is, 28 velos;

therefore in 12 secs. it passes over 28×12 ft., or, 336 ft. (ii)

Example ii. *A particle of* 4 lbs. *which at a certain instant has* 256 velos, *is acted on continuously by a* force *of* 32 poundals *in the direction* opposite to that of the velos; *find* (i) when *and* where *it will have* 224 velos, (ii) when and where *it will have* 224 velos in the opposite direction.

32 poundals acting on 4 lbs. produce in it $\frac{32}{4}$, that is 8, celos.

Hence the particle has -8 celos.

So that, if after x seconds the particle has 224 velos, we have

$$224 = 256 - 8x; \text{ whence, } x = 4.$$

In these 4 seconds the displacement of the particle is 4 times $(256 - 2 \times 8)$ ft. or 960 feet. (i)

If after x seconds the particle has -224 velos,

we have $-224 = 256 - 8x$; whence, $x = 60$.

In these 60 seconds the displacement [Note p. 18] of the particle is 60 times $(256 - 30 \times 8)$ ft. or 960 ft. (ii)

Thus, the particle after 4 seconds has 224 velos; it continues moving in the same direction until the action of the force, producing in it a negative acceleration, brings it to rest; then the action of the force gradually produces in it velocity in the opposite direction, and it will be found that, as it returns, *it repasses each point with the same velocity in the opposite direction.* [See note p. 18.]

35. The force which produces α celos in m lbs. is

$m\alpha$ poundals.

For, to produce 1 celo in mass 1 lb. 1 poundal is required;
to produce 1 celo in mass m lbs. m poundals are required;
to produce α celos in mass m lbs. $m\alpha$ poundals are required.

It is convenient to say that a mass m lbs., which has α celos, has

$m\alpha$ pound-celos.

With this understanding we may say that the number of poundals acting on a mass is equal to the number of pound-celos which are consequently produced.

The **mass-acceleration** of a body is measured by the number of pound-celos which it has.

EXAMPLES. IX.

NOTE. *In these and in all future examples, in which a force is applied to a mass, it is supposed that the force is so applied, that the* shape *of the mass need not be considered; in some cases, the body is supposed to be so* small *that its* size *need not be considered; such a body is called a* **particle.**

Find the acceleration produced in each of the following 6 cases.

1. by 3 poundals acting on 6 lbs.

2. by 32 poundals acting on 1 lb.

3. by 32 poundals acting on 64 lbs.

4. by 64 poundals acting on 5 lbs.

5. by 48 poundals acting on 1 cwt. [= 112 lbs.]

6. by 7 poundals acting on 4 cwt.

Find the velocity produced and the distance passed over in each of the 6 following cases, the mass being *initially at rest.*

7. by 3 poundals acting on 6 lbs. for 10 seconds.

8. by 12 poundals acting on 1 lb. for 4 secs.

9. by 18 poundals acting on 27 lbs. for 3 secs.

10. by 32 poundals acting on 56 lbs. for 9 secs.

11. by 56 poundals acting on 2 cwt. for 12 secs.

12. by 64 poundals acting on 2 lbs. for $\frac{1}{2}$ sec.

13. Find the velocity acquired and the distance passed over in each of the 3 following cases, the force having the same direction as the initial velocity.

(i) 4 poundals acting for 3 secs. on 3 lbs. having initially 4 velos.
(ii) 5 poundals acting for 4 secs. on 10 lbs. having initially 20 velos.
(iii) 22 poundals acting for $3\frac{1}{2}$ secs. on $\frac{1}{2}$ lb. having initially 4 velos.

14. 20 poundals act on a mass of 5 lbs. (initially at rest) for 3 seconds and then cease; how far will the mass go in 10 seconds more?

15. 64 poundals act for 4 seconds on a mass of 2 lbs. initially at rest; after which the force is suddenly reversed; find how far the mass goes in 8 seconds from rest.

16. A particle of 1 lb. which at a certain instant has 96 velos, is acted on by 32 poundals in the direction *opposite* to the velos; when will it be 128 ft. from its position at that instant? and find its velocities at those times.

17. A particle of 1 lb. which at a certain instant has 64 velos, is acted on by 32 poundals in the direction opposite to the velos; when will it be at rest? when will it be moving in the opposite direction with 32 velos?

18. A particle of 1 lb. acted on by a constant force moves in a certain second over 20 ft., and in the next second but one it moves over 128 ft.; find the force.

19. A particle of 1 lb. is at rest, when being acted on by a constant force, it moves over 16 ft. in the first second of its motion; find the force.

20. A mass of 20 lbs. is acted on by a constant force, and in the fifth and seventh seconds of its motion it passes over 108 ft. and 140 ft. respectively: find its initial velocity and the magnitude of the force.

21. A particle acted on by a constant force of 20 poundals passes over 72 ft. while its velocity increases from 16 to 20 velos; find its mass.

22. A particle acted on by 20 poundals attains a velocity of 45 miles per hour after 12 seconds from rest; find its mass.

23. A train of 90 tons on a horizontal plane at rest is acted on by a force of 1000 poundals; the force acts for one minute and then ceases; find how far the train travels before coming again to rest, supposing the friction of the rails etc. to be equivalent to a retarding force of 300 poundals. [N.B. 1 ton = 2240 lbs.]

24. A train of 100 tons attains from rest a velocity of 40 miles per hour in going 1 mile; the breaks are then applied and the train comes to rest in 440 yds. Compare the retarding force exerted by the breaks with the force exerted by the engine (neglecting the rotary motion, and the ordinary friction of the wheels etc.).

WEIGHT.

36. DEF. The **weight** of a mass is the *force* with which it is attracted by the earth.

When we support a mass in our hand so as to keep it at rest, we can only do so by the application of an upward pressure or force; this upward *force* is equal and opposite to the *weight* of the mass.

37. Since the weight of a mass is a force, and since our unit of force is a poundal, in order to measure the weight of a given mass we must express its weight in poundals.

The number of poundals in any force is equal to the number of pound-celos which it produces; accordingly we have to ascertain how many celos the weight of any particular mass produces in that mass.

38. The following experiment is important.

A chamber, such as the receiver of an air-pump, is exhausted of air. In the chamber are placed several masses of different kinds and of different sizes; for instance, a leaden bullet, an iron nail, a bronze penny, a gold sovereign, a glass ball, a piece of cork, a feather; these are all placed on a ledge so arranged that they can be simultaneously let fall.

When a mass is 'let fall' in a vacuum, the *only* force acting upon the mass is its own weight; accordingly in this experiment when the masses are let fall, the weight of each mass produces in its own mass motion, in a vertical direction downwards, which is that of uniformly increasing velocity.

Now it is observed that, having been simultaneously let fall from the same height, these different masses all reach the bottom of the chamber *simultaneously*.

But, when a number of points, starting from rest and each moving in the same direction with uniformly increasing velocity, have equal displacements in the same interval of time, they must all be moving with the *same acceleration*. Hence,

39. The *weight* of any mass is the *force* which when acting alone on that mass, produces in it a certain definite acceleration; which acceleration is the same at the same place on the earth's surface for all masses of whatever magnitude and of whatever kind of material.

This definite acceleration, which is called the **acceleration due to gravity**, is different at different places on the earth's surface.

It is greatest near the Poles (where it is about 32·255... celos), and it gradually diminishes to about 32·091... celos at the equator.

It is the same at places having the same latitude at the same elevation above the level of the sea.

40. It has been ascertained by observation that at Greenwich the number of celos in any mass when *falling freely* towards the earth's surface (that is, when moving *upwards* or *downwards* under the action only of its own weight) is 32·19....

The weight of 1 lb. produces 32·19... pound-celos.

Hence **the weight of 1 lb. = 32·19... poundals.** [Art. 40]

41. It is usual to use the letter g to indicate the *number* of units of acceleration produced in a mass by the action of its own weight.

Hence, at Greenwich $g = 32.19....$

42. The mass-acceleration produced in m lbs. by the *weight* of m lbs. is mg pound-celos.

[$m \times (32{\cdot}19\ldots)$ pound-celos.]

The *force* required to produce mg pound-celos is mg poundals. Therefore

the weight of m lbs. is the *force* **mg poundals.**

43. **N.B.** In all numerical examples we shall use

32 poundals as a sufficient approximation to the **weight** of 1 lb.

Example i. *How many* poundals *are there in* the weight of 3 lbs.?

The weight of 3 lbs. produces in the mass 3 lbs. (about) 32 celos.

Therefore the *weight* of 3 lbs. is equivalent to the *force* 3×32 poundals.

Example ii. *How many* celos *will a force equal to* the weight of 3 lbs. *when applied to a* mass 12 lbs. *produce in it?*

The force is 3×32, or, 96 poundals

and 96 poundals produces in 12 lbs., $\frac{96}{12}$ or 8 celos.

Example iii. 16 lbs. *is acted on continuously by a force equal to the weight of* 4 lbs.; *find its* acceleration; *and find how far it moves from rest in* 10 seconds.

The weight of 4 lbs. $= 4 \times 32$ poundals.

This force 4×32 poundals acting on 16 lbs. produces in it $\frac{4 \times 32}{16}$ celos;

That is, this force produces in 16 lbs. the acceleration 8 celos.

In 10 seconds the 16 lbs. will pass over a distance s feet such that

$$s = \tfrac{1}{2}at^2 = \tfrac{1}{2} \times 8 \times 100 = 400.$$

The distance required is 400 feet.

NOTE. A force is said to be expressed *in pounds weight*, when we state the number of lbs. whose *weight* is equal to the force.

Example iv. A mass 4 cwt. *is observed to be moving with* 16 celos; *express* in pounds weight *the* force *which is acting upon it.*

When 4 cwt. has 16 celos, it has $4 \times 112 \times 16$ pound-celos.

To produce this, $4 \times 112 \times 16$ poundals must be acting upon it;

$4 \times 112 \times 16$ poundals are equal to the weight of $\frac{4 \times 112 \times 16}{32}$ lbs.;

that is, the weight of 2 cwt. must be acting on the 4 cwt.

EXAMPLES. X.

Express the four following forces in poundals:

1. The weight of 3 lbs. **2.** The weight of half an ounce

3. The weight of 20 tons. **4.** The weight of 4 cwt.

Express the six following forces in lbs. weight:

5. 128 poundals. **6.** 1000 poundals.

7. The force which produces 32 celos in 100 lbs.

8. The force which produces 4 celos in 1 cwt.

9. The force which acting on 1000 lbs. will produce in it 20 velos in 5 secs. from rest.

10. The force which is acting on a mass of 4 cwt. which moves over 64 ft. from rest in 2 seconds.

11. 1 cwt. placed on a smooth horizontal plane is acted on continuously by a horizontal force equal to the weight of 7 lbs.; find its acceleration.

12. 3 tons placed on a smooth horizontal plane is acted on continuously by a horizontal force of 42 lbs. weight; how long will it take to gain from rest a velocity of 15 miles an hour?

13. 28 lbs. placed on perfectly smooth ice is acted on by a horizontal force equivalent to the weight of 1 lb.; find its velocity after 7 minutes.

14. A train of 100 tons is drawn with an acceleration of 4 celos; express in pounds weight the force exerted by the engine in addition to that necessary to overcome the friction.

15. 1 ton on a smooth horizontal plane is observed to move from rest a distance of 48 ft. in 60 seconds; if the motion is caused by a uniform horizontal force, how many lbs. would the force lift?

16. A train of 100 tons is drawn by an engine which can exert a force, which would lift a weight of 800 lbs.; supposing the force of the friction etc. to be equivalent to a force of 100 lbs. weight, find how long the train would be in attaining from rest a velocity of 60 miles per hour.

17. A mass of 1 cwt. is placed on ice and is acted on by a horizontal force equal to the weight of 8 lbs.; supposing that a force equal to the weight of 1 lb. is sufficient to overcome the friction of the ice, find when the mass will have 100 velos.

18. Supposing the force of 8 lbs. weight of Question **17** to be withdrawn when the mass is moving at the rate of 60 miles an hour, in how many seconds will the mass be brought to rest by the retardation due to the friction?

19. A train is acted on by a horizontal force (over and above that necessary to overcome the friction) equal to the weight of 10 cwt.; it is observed to go 20 ft. from rest in 30 seconds; find the mass of the train.

20. A lump of chalk is placed at rest on a smooth horizonal plane and is acted on continuously for 10 seconds by a force equivalent to the weight of 2 lbs.; the force is then withdrawn and after that the mass is observed to move 100 yards in half a minute. What is the mass of the chalk?

44. The motion of a mass falling vertically (either *upwards* or *downwards*) under the action of its own weight only is that of a point moving with 32 celos downwards.

Example†. *How long does a stone take in falling from rest under the action of gravity through a vertical distance of* 128 ft.?

The stone has an acceleration of 32 celos downwards.

Let t seconds be the time of falling 128 ft. from rest;

[Since $s=ut+\frac{1}{2}at^2$, and $u=0$]

then, $128=\frac{1}{2}\times 32t^2$;

or, $t^2=8$;

therefore, $t=2\sqrt{2}=2\times 1\cdot 4142=2\cdot 8284\ldots$

The interval occupied is therefore 2·8...seconds.

EXAMPLES. XI.

1. A stone is let fall from the top of a tower 256 ft. high; in how many seconds will it reach the ground?

2. A stone is let fall; how far will it fall (i) in the 2nd second, (ii) in the 4th second from rest?

3. A stone is thrown vertically upwards with a velocity of 320 velos; when will it come to rest? and how high will it go?

4. A stone is thrown vertically upwards and returns after 4 seconds to the point from which it was thrown; with what velocity was it thrown? and how high did it go?

5. A stone is thrown vertically upwards and passes a point 48 ft. high with a velocity of 32 ft. per second; in how many seconds will it return to the point of projection?

6. A stone is let fall from a tower 256 ft. high and at the same instant another stone is projected from the foot of the tower with just sufficient velocity to carry it to the top of the tower; when and where will the stones meet?

45. In Art. 28 it is assumed that we can apply a *constant* force to a mass.

Such a force might be obtained by taking care that the force is just sufficient to keep a certain steel spring compressed in a certain manner.

† Example iv. p. 15 and Example ii. p. 21 should be noticed.

By applying the same constant force to each of two bodies in turn and observing the number of celos produced in them, we obtain a measure of the mass of the bodies.

46. It happens however that the experiment of Art. 38 provides a more convenient method of measuring the mass of a body. For, since the weight of a mass is a force which produces a fixed number (g) of celos in the mass itself, the *weight* of a mass is proportional to the mass itself; therefore, **equal masses have equal weights.**

By a *ton* of coal is meant a certain amount of mass.

That a certain quantity of coal contains a ton mass is ascertained by weighing it; that is, we ascertain that the mass of coal has the same weight as a certain other standard mass of iron, which mass we call a ton.

47. In order to understand what is meant by the statement '*equal masses have equal weights*,' let us consider what would be the practical result if two equal masses had not equal weights.

Imagine for example that we have two lumps, one of iron, the other of brass, which are of equal mass but have very different weights. That is, suppose the Earth attracts one of the masses, say the iron, very strongly, the brass very slightly.

The *masses* are equal; let them be placed on a smooth horizontal plane; apply equal horizontal forces with a steel spring or with the hand; they will each have the same acceleration. Bring them to rest again and lift them vertically; the iron would require a much larger force, to support it when lifted, than the brass. Or again, place them in the opposite pans of a weighing machine; then one of the pans would be depressed with a strong force.

NOTE. In the language in common use, the ideas of *mass* and *weight* are not clearly distinguished.

For example, a piece of metal is said to be

'*a* 7 lbs. *weight*';

these words mean, a piece of metal which is '*a mass of* 7 lbs. *whose* weight *is used for the purpose of comparing the* weights *of other masses.*'

Or again, 'a train *weighing* 100 tons';

here, the word *weighing* means, '*whose* weight *is equal to that of*' 100 tons.

Again, a string is often said to be *light*, meaning *weightless*, or *of too small* **weight** *to be observed;* such a string must also *be of too small* **mass** *to be observed.*

The student however must carefully remember that

the **unit mass** is 1 lb.,

while **the weight of 1 lb. is 32·2...poundals.**

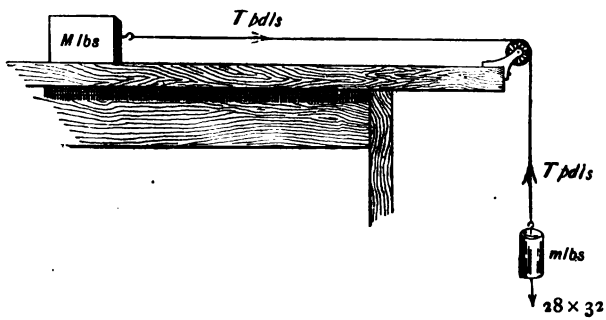
48. The following example should be noticed.

Example. A mass of 1 ton *lies on a smooth horizontal table, and is attached by a light string which passes over the edge of the table, to a mass of* 28 lbs., *which hangs freely under the action of its own* weight; *find the tension of the string.*

[NOTE: Here, the force acting on the ton mass is **not** equal to the weight of 28 lbs.; for

The force acting on the ton is the pull or *tension* of the string; and the tension of the string is *not* equal to the weight of the 28 lbs.; for if it were, the 28 lbs. mass would have acting on it (i) a pull equal to the weight of 28 lbs. *upwards* and (ii) its own weight (which is that of 28 lbs.) *downwards;* that is, it would have no *resultant* external force acting on it, and would therefore have *no acceleration.*

But, if the ton has acceleration, the 28 lbs. must have acceleration also. Hence, the force acting on the ton, i.e. the tension of the string, is *less* than the weight of 28 lbs.; as will presently appear.]



I. Consider the external forces acting on the 28 lbs.†

Let T poundals be the tension of the string; then there are two external forces acting on the mass of 28 lbs., its weight directly downwards, and the pull of the string directly upwards,

$\therefore$ the resultant force is 28×32 poundals $- T$ poundals.

Let this resultant force produce a celos in the 28 lbs.

It produces therefore $28a$ pound-celos;
the force necessary to do this is $28a$ poundals; hence we see that

$$(28 \times 32 - T) \text{ poundals} = 28a \text{ poundals.} \qquad \text{(i).}$$

† The student must learn to fix his attention exclusively on *one mass only*, forgetting for the time the existence of the other mass.

II. Consider the external force acting on the ton.

The *weight* of the ton does not concern the question; it is supposed to be entirely obliterated by the upward support of the smooth horizontal plane.

The only other force acting on the ton is the tension of the string. The string is supposed to be *light*, and therefore of no mass; so that it exerts the same pull on the ton as it does on the 28 lbs.; which pull is

T poundals.

Also, since the two masses are connected by an *inextensible* string, the acceleration of the ton is the same as that of the 28 lbs. viz., α celos.

The force necessary to produce α celos in 2240 lbs. is

2240α poundals;

but, the force which produces the α celos is T poundals; hence we see that T poundals $= 20 \times 112\alpha$ poundals $= 80 \times 28\alpha$ poundals; (ii)

but from I, $(28 \times 32 - T)$ poundals $= 28\alpha$ poundals; (i)

whence, adding (ii) to (i) $28 \times 32 = 81 \times 28\alpha$;

therefore, $\alpha = \frac{8}{9}\frac{8}{1}$.

Now, $T = 20 \times 112\alpha = 20 \times 112 \times \frac{8}{9}\frac{8}{1}$,

$\therefore$ the tension = the weight of $20 \times 112 \times \frac{8}{9}\frac{8}{1} \times \frac{1}{3}\frac{1}{2}$ lbs.

= weight of $27\frac{8}{8}\frac{8}{1}$ lbs.

49. DEF. The **inertia** or **inertness** of *mass* is that essential property by virtue of which mass requires the application of a finite external force for a definite interval of time in order to produce a finite change in its velocity.

Consequently it is by virtue of its inertia that mass persistently retains its existing velocity unaltered, so long as it is not acted on by external force.

The nature of *force* is such that it needs *time* in which to change the velocity of mass.

Mass is **inert**, in the sense that it cannot of itself make any *change* in its velocity.

External force acting on mass produces in it acceleration only; it does *not* produce any definite *velocity* unless it acts *continuously* for some definite interval of time.

Example. When it is observed that a given mass M has moved in a given interval from a point A to another B, we can from this infer nothing as to whether an external force has acted on the mass during that interval or not. The mass M may have moved over that distance by reason of its inertia; by virtue of which it persisted in moving from A to B with uniform velocity.

But when it is observed that a mass M has at one instant a certain *velocity* and that after some interval it has a different velocity, we are able to state that, for some part at least of that interval, it has been acted on by some external force.

In Art. 48, the reason why the acceleration of the ton in the Example is less than that which would be produced by the force 28×32 poundals is that the weight of 28 lbs. has to overcome the '*inertia*' of (1 ton + 28 lbs.).

EXAMPLES. XII.

1. A sledge of 1 ton is pulled on ice by a rope whose tension is equal to the weight of 56 lbs.; if the friction of the ice causes a horizontal retarding force on the sledge equal to $\frac{1}{80}$ of its weight, find the acceleration produced.

2. A sledge of 1 ton is pulled on ice by 4 dogs whose weight is 28 lbs. each; each dog obtains from the friction of the ice a horizontal force forwards equivalent to half its own weight; prove that if the friction of the sledge on the ice is $\frac{1}{80}$ of its weight the acceleration produced is $\frac{16}{105}$ celos.

3. A man of 12 stone on a railway truck weighing $1\frac{1}{2}$ tons propels it on a smooth horizontal railway, when he is himself on it, by means of a rope fastened to a fixed point; he pulls with a constant force of 40 lbs. weight; compare the acceleration of the truck with that which he would produce if, fastening the rope to the truck and getting off, he pulled at the rope with a force of 40 lbs. weight.

4. A mass of $1\frac{1}{2}$ tons placed on a smooth horizontal table is acted on by a horizontal force of 40 lbs. weight; find how far it would move from rest in 2 seconds.

5. A mass of $1\frac{1}{2}$ tons lies on a smooth horizontal table and is attached by a light string, which passes over the edge of the table, to a mass of 28 lbs. which hangs freely under the action of its own weight; how far will the masses move from rest in 2 seconds?

6. A man of 12 stone finds that he can in running attain a speed of 15 miles an hour in 3 seconds; find the horizontal force which he applies to his own mass, supposing that he attains that speed with uniform acceleration.

CHAPTER III.

STRESS.

50. When two **equal forces** act *simultaneously* throughout the same interval each on one of two masses M and M', they produce in each mass the *same number of pound-velos.*

For at any instant one force is producing in the mass on which it acts the same number of pound-celos which the other force is at that instant producing in the mass on which it acts. Therefore in the same interval each force produces the same number of pound-velos.

The forces are not necessarily *uniform*, provided they are always *equal;* one force may be changed provided the other force is simultaneously changed also.

When the *masses* M and M' are not equal, the *accelerations* produced in them by these equal forces will be unequal; and therefore the *velocities* added to the masses respectively in the course of any interval will be unequal.

51. The action of two exactly equal forces, acting simultaneously for the same interval of time on different masses, in exactly opposite directions, is a phenomenon which is constantly taking place.

52. No external force can be applied to one mass by another, without the first mass exerting an equal and opposite force of the mass which applies the force. This is

Newton's Third Law. '*To every action there corresponds an equal and opposite reaction.*' [See page 57.]

53. DEF. **A stress** is that which consists of two *equal* and *opposite* forces together forming an action and its reaction.

Examples. The *pressure* of a heavy mass on the ground on which it rests, and the pressure of the ground on the mass, together form a stress.

The *tension* of a string at any point of its length is a stress.

54. When two particles act and react the one on the other, the stress causes no alteration in the *total* mass-velocity of the two particles; that is, the stress has no effect on the algebraical *sum* of their mass-velocities.

For a stress consists of two equal and opposite forces acting simultaneously; so that [Art. 50] whatever amount of mass-velocity the action produces in one mass in any interval, the reaction produces in the other mass in the same interval an exactly equal amount in the opposite direction.

55. When the two masses m lbs. and m' lbs. are so connected that their velocities are always equal, since the stress set up between them has no effect on the sum of their mass velocities, [that is, upon $(m + m') \times v$,] it follows that the stress has no effect [upon v, that is,] upon their velocity.

DEF. When two particles are considered as a single mass the stress between them is said to be **internal.**

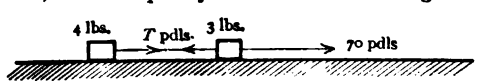
Hence the above result may be stated thus,

An *internal stress* between two particles (which cannot move relatively to each other) *has no effect on their joint velocity.*

Example i. The stress communicated by a light inextensible string from a mass m lbs. to another mass m' lbs., has no effect in altering their joint velocity. Provided the string is always kept tight they may be treated as one mass, and the tension of the string need not be considered in discussing their joint velocity.

Example ii. *Two masses* 3 lbs. *and* 4 lbs. *placed on a smooth horizontal plane are connected by a light inextensible string; a force of* 70 poundals *is applied to the mass* 3 lbs. *in the direction of the string, which is tight; find the* acceleration *produced; also the* tension *of the string.*

I. The masses are prevented by the string from moving relatively to each other; hence, we may consider the tension of the string as an internal stress, and consequently the two masses as a single mass.



Thus, we have a mass of $(3+4)$ lbs. acted on by 70 poundals; therefore the acceleration α is given by the equation $70 = (3+4)\alpha$;

that is, the acceleration is 10 celos (i).

II. To find the *tension* of the string we must consider each component of the stress separately.

Let the string apply an external force $-T$ poundals to the mass 3 lbs.; it will consequently apply an external force T poundals to the mass 4 lbs.

Then, if a celos be the acceleration (which is the same for each mass, for the string is tight and inextensible)

we have, $70 - T = 3a$; [from the motion of the 3 lbs.

and, $T = 4a$; [from the motion of the 4 lbs.

whence, by addition, $70 = (3+4)\,a$ (i),

wherefore $T = 40$.

The tension required is 40 poundals (ii).

EXAMPLES. XIII.

1. An engine of 40 tons is attached to a train of 6 carriages of 10 tons each; the train is moving with 5 celos; neglecting friction and the rotatory motion of the wheels, find the tension of the coupling between the engine and the next carriage.

2. In the train in Question **1** find the tension of the coupling between each pair of carriages.

3. In the train in Question **1**, what force is the engine exerting?

4. The train in Question **1** moving at the rate of 30 miles an hour is stopped by brakes on the engine in a quarter of a mile; find

(i) the force exerted by the brake,

(ii) the stress between the engine and the next carriage,

(iii) the stress between the two last carriages,

(iv) if the brakes on the engine exerted the same force in all cases, how far would the engine alone, going 30 miles an hour when the brakes are applied, run before coming to rest?

5. An engine which exerts a horizontal force sufficient to generate an acceleration a celos in its own mass m lbs., is attached to 3 carriages whose masses are $\frac{1}{2}m$ lbs., $\frac{1}{4}m$ lbs. and $\frac{1}{8}m$ lbs. respectively; find the acceleration of the train and the stress between each carriage, neglecting friction and the rotary motion of the wheels.

6. An engine which exerts a horizontal force p poundals, sufficient to generate in its own mass a celos, is attached to n carriages each of which is $\frac{1}{10}$th of the mass of the engine; find the acceleration of the train and the stress between the 4th and 5th carriages, neglecting friction etc.

56. The ideas of mass, force, stress, etc. may be illustrated and tested experimentally by the aid of

Atwood's machine.

Atwood's machine may be described as follows.

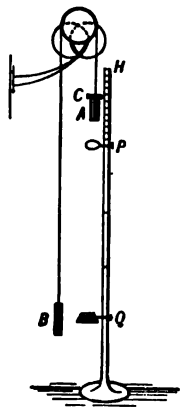
A wheel or pulley, made as light and as free from friction as possible, is fixed at a convenient height from the ground; over the pulley is passed a light inextensible string, usually made of silk; from the ends of the string are suspended two masses M and M'. Each mass is acted on only by its own weight and by tension of the string.

I. It will be found that we can arrange the masses so that the *difference* between the tension of the string and the weight of either mass is *small* compared with the weight of either mass; and it follows, that in this case, each mass has an acceleration which is *small* compared with that produced in a mass falling freely under the action of its own weight. In fact the acceleration can be made so small that the velocity can be observed and can be shewn by actual measurement to be *uniformly increasing*. We can thus test the statement—that force produces in mass that kind of motion which has been defined as uniform acceleration.

II. We can test by observation the statement that mass when acted on by no resultant external force moves with uniform velocity.

III. Since the mass of the string is very small, it causes no appreciable effect on the tension; if we also neglect the effects produced by the mass of the wheel and the friction of the axle (which are both comparatively small), then the tension remains unchanged throughout the whole length of the string. The string may then be described as *the means of communicating* the two portions of a stress which act one on each of the masses; in this way, by comparing the actual motion of the masses with the motions given by our calculations, we have a practical test of the truth of the Third Law of Motion. [See page 57.]

Example i. *In an* Atwood's machine *the masses* A *and* B *are each* $15\frac{1}{2}$ oz. *and a mass* C *of* 1 oz. *is placed gently on the mass* A *when at rest; after* 2 seconds C *is removed; discuss the resulting motion.*



I. Consider the motion of the masses A and C, considering them as one mass.

The external forces are (the weight), $\frac{33}{32} \times 32$ poundals, and (the tension) $-T$ poundals *downwards;* let a celos *downwards* be the acceleration, the mass is $\frac{33}{32}$ lbs.; the mass-acceleration is $\frac{33}{32} \times a$ pound-celos; whence, as in Ex. i,

$$33 - T = \tfrac{33}{32} a \qquad \text{(i)}.$$

II. Consider the motion of the mass B; then the forces are $\frac{31}{32} \times 32$ poundals, and $-T$ poundals, *downwards;* the mass is $\frac{31}{32}$ lbs., the acceleration is $-a$ celos *downwards;* therefore, as in Example i, we have

$$31 - T = -\tfrac{31}{32} a \qquad \text{(ii)}.$$

Whence by subtraction

$$2 = \tfrac{64}{32} a = 2a,$$

or

$$a = 1 \qquad \text{(iii)}.$$

Thus, as long as the mass C forms part of the mass A each mass has 1 celo, the mass B upwards and the masses A and C downwards.

Hence at the end of 2 seconds each mass will have 2 velos and will have moved over $(\frac{1}{2}at^2)$, that is, 2 feet.

III. Forming the equations (i) and (ii) on the supposition that C has been removed from A, we have

$$31 - T' = \tfrac{31}{32} a \quad [T' \text{ here is not the same as the } T \text{ above.}] \qquad \text{(i)}$$

and,

$$31 - T' = -\tfrac{31}{32} \qquad \text{(ii)}$$

whence

$$a = 0.$$

So that when, at the end of 2 seconds, the 1 oz. mass is removed, the acceleration vanishes; that is, the velocity is uniform; and the masses each move with 2 velos, A downwards and B upwards.

Example ii. *Find the* stress *between the small* mass C *and the* mass A *in the above Example* (i).

The mass A and the mass C each descends with 1 celo.

The weight of the mass C, viz. (of $\frac{1}{16}$ lb.) 2 poundals acting on $\frac{1}{16}$ lb. would produce in it 32 celos. It actually has 1 celo.

Hence the stress is such that the upward part of it acting on $\frac{1}{16}$ lb. destroys in it 31 celos.

The upward action of the stress is therefore $\frac{31}{16}$ poundals.

Example iii. *A horizontal platform is made to descend vertically with* 4 celos; *a man stands on the platform with* 56 lbs. *in his hand; shew that this mass appears to weigh only* 49 pounds.

The force acting on the 56 lbs. (besides the pressure of the hand) is its weight downwards, that is, 56×32 poundals.

This would produce in it 32 celos, if the upward force of the man's hand did not prevent. The 56 lbs. actually has 4 celos, which requires

56×4 poundals to produce it.

Thus, of the weight (56×32 poundals) the part 56×4 poundals is taken up in producing the 4 celos; so that the upward force of the man's hand must counteract the remainder. In other words this force is

56×28 poundals; that is, the weight of 49 lbs.

57. To find the acceleration in an Atwood's machine we may proceed as follows.

Let the masses be m lbs. and m' lbs.

Then, considering the string as the means of preventing relative motion between the masses, and therefore the mass moved to be the sum of the masses m lbs., and m' lbs., the tension of the string becomes an internal stress and has no effect on the mass-acceleration produced.

The resultant external force in the direction of the motion of the particles is (the weight of the greater mass m – that of the less m'), hence we have $mg - m'g = (m + m')\, a$.

58. The problem of Atwood's machine is similar to the following.

Two particles A and B of masses m lbs. and m' lbs., connected by a light inextensible string, are placed with the string stretched, on a smooth horizontal plane; A is acted on by a horizontal force mg poundals in the direction BA, and B is acted on by a horizontal force $m'g$ poundals in the direction AB; find the tension of the string and the acceleration of the masses. It will be found [Art. 55] that the tension is

$\frac{2mm'}{m+m'}g$ poundals; and that the acceleration is $\frac{m-m'}{m+m'}g$ celos.

EXAMPLES. XIV.

Find the tension of the string and the acceleration of the masses in an Atwood's machine, when the masses A and B have the following values:

1. Mass of A, 1 lb. 1 oz.; of B, 15 oz.
2. Mass of A, 5 lbs.; of B, 3 lbs.
3. Mass of A, 25 oz.; of B, 23 oz.
4. Mass of A, 4 lbs. 4 oz.; of B, 3 lbs. 12 oz.

5. In an Atwood's machine the masses A and B are 3 lbs. 15 oz. each; a mass of 2 oz. is gently placed on A and is removed by a catch after A has descended 1 ft.; shew that A will take 4 seconds to descend the next 4 ft.

6. In an Atwood's machine the masses are each 7 lbs. 15 oz.; a mass of 2 oz. is gently placed on one of them and is removed by a catch after descending 8 ft.; how long will it take to descend the next foot?

7. In an Atwood's machine it is observed that after descending 8 ft. from rest the masses have a velocity of 4 ft. per second; the larger mass is $32\frac{1}{2}$ oz.; find the other mass.

8. In an Atwood's machine the masses are equal and it is observed that a weight of 1 oz. added to one of them produces 1 celo; what are the masses?

9. In an Atwood's machine the smaller mass is 10 lbs.; what is the larger when the acceleration is 3 celos?

10. In an Atwood's machine the acceleration is 2 celos; what is the ratio of the weights?

11. Prove that in an Atwood's machine when the tension of the string is one-fourth of the sum of the weights $m : m' = 3 + 2\sqrt{2} : 1$.

12. Two masses m lbs. and m' lbs. connected by an inextensible thread, are placed with the thread straight on a smooth horizontal plane; the mass m is acted on in the direction of the thread away from m' by a force f poundals, and m' is acted on by a force f' poundals in the opposite direction; find the acceleration of the masses.

13. Prove that if in Ex. **12** the forces are equal to the weights of the masses on which they act, the acceleration is $\frac{m-m'}{m+m'}g$ celos.

14. A bullet fired directly into a block of wood will penetrate 3 in.; what part of its velocity will it lose in passing through a board of the same wood 1 in. thick supposing the resistance uniform?

15. A mass of 1 cwt. is placed on a passenger lift which descends vertically with a uniform acceleration of 10 celos; find the pressure between the lift and the mass.

16. If the lift in Question **15** is ascending vertically with an acceleration 16 celos, what is then the pressure?

17. A man can raise a sack of corn weighing $1\frac{1}{2}$ cwt.; shew that when on a lift which is made to descend vertically with 8 celos, he can raise 2 cwt.

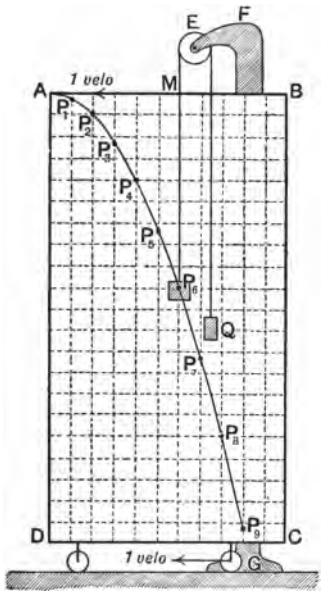
18. A balloon ascends vertically with a uniformly accelerated motion so that a weight of 1 lb. produces, on the hand of the aeronaut sustaining it, a downward pressure equal to that which 17 oz. produces when at rest; what height does the balloon attain in 1 min. from rest?

19. Find the pressure of the mass of 2 oz. on A in Question **5**.

20. Find the magnitude of the stress between the 2 oz. and the mass on which it is placed in Question **6**.

NOTE. By the following arrangement we may exhibit pictorially the motion of a particle under the action of a constant acceleration.

Let a screen *ABCD* be arranged so that it can be moved horizontally with uniform velocity in front of an Atwood's machine *GEF*.



Let the heavier of the two weights carry a soft pencil, and let it be so arranged that as the weight descends the pencil leaves a mark on the screen.

Let a series of vertical lines be drawn on the screen *ABCD* at equal distances from each other as in the figure.

Then as the screen moves with uniform velocity the passage of the pencil across each vertical line will mark the end of equal intervals of time.

If a series of horizontal lines at equal distances be also ruled upon the screen the whole motion is rendered clear pictorially.

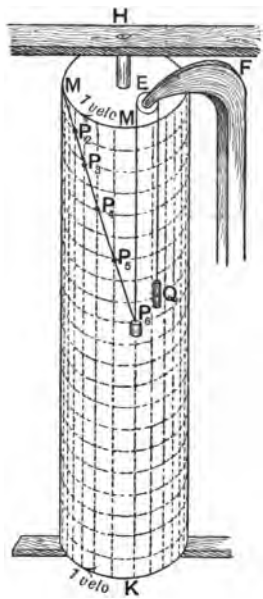
Example. Let the weights P, Q in an Atwood's machine be so arranged that the heavier weight P has $\frac{1}{2}$ celo downwards.

Let the interval between consecutive lines, both vertical and horizontal, on the screen be 1 foot. Let P be initially at A on the screen and be held at rest, and then let go while the screen has the velocity 1 velo. Then the weight P descends $\frac{1}{2} \times \frac{1}{2} . t^2$ feet in t seconds. Thus, as P crosses the first vertical line (i.e. at the end of 1 second) it has descended $\frac{1}{4}$ ft. and the pencil is at P_1 on the screen, as P crosses the second vertical line it has descended 1 foot and the pencil is at P_2 on the screen, as P crosses the third vertical line it has descended $2\frac{1}{4}$ feet and the pencil is at P_3 on the screen and so on.

Morin's Machine is a convenient arrangement of the apparatus described above.

The screen is wrapped round a vertical cylinder which is so arranged, that by means of clock-work it can be made to revolve uniformly about its axis *HK*.

The weight *P* then descends vertically in front of the cylinder.



The convenience of the machine consists chiefly in the absence of any necessity to start the weight *P* at any particular instant. The cylinder revolves uniformly and the mark of the pencil on the screen indicates the instant at which the weight was let go.

In the machine actually constructed by M. Morin the weight *P* descended freely being guided by an upright slot so that its acceleration was (neglecting resistances) the acceleration due to gravity. The machine unless on a colossal scale could only give the trace for about 1 second.

EXAMPLES. XIV a.

1. Suppose in the figure of page 46 the screen has 2 velos and the weight $\frac{1}{2}$ celo, draw the mark on the screen made during the first 8 seconds to the scale of $\frac{1}{4}$ inch to a foot.

2. Suppose that the screen has 1 velo and the weight 1 celo, draw the path for the first 8 seconds on the scale of $\frac{1}{8}$ inch to 1 foot.

3. Draw the trace of the pencil on the screen in Morin's machine for the first 8 seconds when the surface of the cylinder moves with 64 velos and the weight descends with 32 celos on the scale of $\frac{1}{512}$ inch to a foot.

CHAPTER IV.

IMPULSE. IMPACT. ELASTICITY.

59. By Art. 33 *Mass* never changes its uniform velocity in a straight line, except it be acted on by *continuous force.*

By Art. 33 *Force* produces *acceleration* in the mass on which it acts.

Force is measured by the *mass-acceleration* which it produces [see Art. 35].

Therefore **Mass** is such that it **cannot** *instantaneously* receive a finite addition to its velocity.

For by Art. 18 acceleration signifies a *growth* of velocity, and an acceleration *always* requires an interval of time in which to produce a finite change of velocity.

Hence, *Mass* [see Art. 33] is such that in order to produce a finite change in its velocity, two conditions are necessary; **a continuous force** must act upon it, **and an interval of time** must elapse.

60. DEF. The **Mass-velocity** [or **Momentum**] of a body is measured by the product of the number of pounds in the body by the number of velos which it has.

A body of 1 lb. has **unit** *mass-velocity*, when it has 1 velo; it is then said to have a **pound-velo.**

DEF. An **impulse** is that which produces in the mass on which it acts a finite velocity, so that the *mass-velocity* produced varies as the impulse producing it.

61. DEF. We choose as our **unit** *impulse* that impulse which acting on a pound produces in it 1 velo.

We shall call the *unit impulse* a **pulse.**

Hence the number of *pulses* in an impulse is equal to the number of *pound-velos* which it produces, so that Art. 60 may be stated thus

DEF. An *impulse* is numerically equal to and is in the same direction as the *mass-velocity* which it produces.

62. Since an impulse produces a finite change in the velocity of mass, *an impulse* is by Art. 59 the joint effect of a continuous force and of an interval of time.

In what follows we shall only consider those impulses which are produced by *constant* force.

1 pulse has the same effect on a mass as 1 poundal acting continuously throughout 1 second; each produces a pound-velo in the mass on which it has acted. It has the same effect as (i) 2 poundals acting for $\frac{1}{2}$ second;

as (ii) 100 poundals acting for $\frac{1}{100}$th of a second;

as (iii) 10000 poundals acting for $\frac{1}{10000}$th of a second;

and so on.

A *pulse* might be termed a *poundal-second;* meaning that a pulse has the same effect as a poundal acting continuously for a second.

A *pulse* = a *pound-velo* = a *poundal-second.*

63. **The force of an impulse** is the force (here supposed constant) whose continued action for some interval produces the impulse.

The interval of an impulse is the interval throughout which the action of the force is continued to make the impulse.

Example 1. *A* cricket ball *of* $\frac{1}{2}$ lb. *receives an additional velocity of* 100 velos; *what* impulse *has been applied to it?*

Its mass velocity is increased by $\frac{1}{2} \times 100$ pound-velos,

$\therefore$ 50 pulses have been applied to it.

Example ii. *If in* Example i. *the* interval *of the impulse were* (i) $\frac{1}{2}$ of a second, (ii) $\frac{1}{10}$th of a second, (iii) $\frac{1}{1000}$ of a second, *find the* force of the impulse (*supposed uniform*) *in each case.*

(i) The acceleration necessary to produce 100 velos in $\frac{1}{2}$ second is 200 celos. Hence the acceleration of the cricket ball in this case was 200 celos. The mass is $\frac{1}{2}$ lb.,

$\therefore$ the force is $\frac{1}{2} \times 200$ poundals; or 100 poundals $= 3\frac{1}{8}$ pounds weight.

(ii) The acceleration necessary to produce 100 velos in $\frac{1}{10}$ sec. is 1000 celos,

$\therefore$ the force is 500 poundals; or $15\frac{5}{8}$ pounds weight.

(iii) The acceleration necessary to produce 100 velos in $\frac{1}{1000}$ sec. is 100000 celos,

$\therefore$ the force of the impulse is 50000 poundals; or $1562\frac{1}{2}$ pounds weight.

EXAMPLES. XV.

1. A train of 100 tons is observed to be moving on smooth horizontal rails with a velocity of 15 miles an hour; after an interval it is observed to be moving with a velocity of 45 miles an hour; what impulse has been applied to it?

2. If in Question **1** the interval is 5 minutes and a uniform force is acting on the train, what is the force?

3. If in Question **1** a uniform horizontal force (beyond what is necessary to overcome friction, etc.) of 28 lbs. weight is acting on the train, what is the interval?

4. An ironclad war ship of 10000 tons is moving with a velocity of 18 miles an hour, and after an interval it is at rest, what impulse has acted upon it? also supposing the interval to have been 5 minutes, what average force (beyond that necessary to overcome friction etc.) has been acting upon it?

5. If in Question **4** the interval were 10 seconds; what would be the force?

6. An ironclad of 8000 tons attains from rest a velocity of 15 miles an hour; what is the impulse necessary? If the interval required is 5 minutes, what is the average resultant force (beyond that necessary to overcome friction etc.) which its engines can obtain from the blades of the screws?

7. A railway engine of 50 tons impinges on a sandbank with a velocity of 60 miles an hour; what impulse does it apply to the sandbank? If it is brought to rest in $\frac{1}{2}$ a second, what pressure does it apply to the sandbank?

8. A football of $\frac{1}{2}$ lb. is moving just after it has been kicked from rest with 20 velos; what impulse has it received? and find the force of the impulse supposing the interval of the impulse to have been (i) $\frac{1}{2}$ sec., (ii) $\frac{1}{10}$ sec., (iii) $\frac{1}{1000}$ sec.

*IMPACT.

64. When one mass meets another mass having a different velocity the masses are said to **impinge.**

The meeting of the two masses is called an **impact.**

The impact is said to be **direct** when the masses are moving in the same straight line, provided the direction of the stress caused by the impact is also in that straight line.

Let us consider the case of one mass having a certain velocity impinging on another mass having a different velocity, the masses being each composed of a hard substance, such as a metal or glass.

In one or both of the masses, since the substances can yield only very slightly at the points in contact, there must be a very rapid change of velocity; that is, there must be a very great acceleration; consequently there must be an immense *mutual pressure* between the masses.

In every such case the impact must, by Art. 59, occupy some definite interval of time.

The particles of each mass yield slightly in the neighbourhood of the great pressure, and then, either return more or less exactly to their original position (in which case the material is said to be **elastic**), or do not return (in which case the impact causes a permanent indentation, and the material is said to be **inelastic**).

65. An impact between two masses causes a stress to be set up between them; which stress lasts for a small interval. The action and reaction of this stress produce *equal* external impulses which act on the two masses in opposite directions.

These equal and opposite impulses produce equal and opposite mass-velocities, one in one mass, the other in the other. Hence, the algebraic sum of the *additional* mass velocities produced during an impact is zero; in other words,

when two masses impinge their total mass-velocity is unaltered by the impact. [See p. 40.]

66. DEF. Two masses are said to be **inelastic** when, after direct impact, their velocities after impact are equal in every respect.

NOTE. When one mass **sticks** to another on impact; or when one mass **penetrates** another, as a bullet penetrates a piece of wood, and stays in it; in fact in every case in which the masses after impact *move as one mass*, the masses are said to be *inelastic*.

Example i. *A mass* M *of* 3 lbs. *having* 5 velos *impinges directly on a mass* M′ *of* 7 lbs. *at rest; find their subsequent motion, the masses being* inelastic.

Since the masses are inelastic, their velocities are made equal by the impact; let their velocity after impact be v velos.

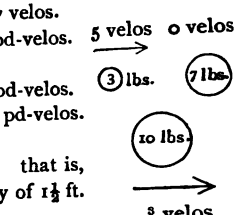
Before impact { the mass-vel. of M is 3×5 pd-velos. / the mass-vel. of M' is 0.

After impact { the mass-vel. of M is $3 \times v$ pd-velos. / the mass-vel. of M' is $7 \times v$ pd-velos.

Hence, by Art. 65, $3v + 7v = 15$;

or, $v = \frac{3}{2}$; that is,

after impact they move together with a velocity of $1\frac{1}{2}$ ft. per second.



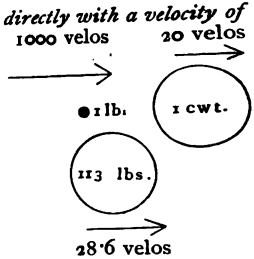
Example ii. *A leaden ball of* 1 lb. *impinges directly with a velocity of* 1000 ft. per sec. *on a mass of* 1 cwt. *moving in the same direction with a velocity of* 20 ft. per sec.; *find their subsequent velocity.*

Let them have v velos after the impact; then, since the total mass-velocity is unchanged

$$1 \times 1000 + 112 \times 20 = v \times (112 + 1),$$

or, $v = \frac{3240}{113} = 28{\cdot}6\ldots$

Thus the masses move together with a velocity of 28·6... ft. per second.



67. It should be noticed [Art. 62] that the smaller the *interval* of a given impulse, the larger is the average *force* which produces that impulse.

Also, the smaller the *interval* (other circumstances being equal), the smaller is the *distance* passed over by the mass during the impulse.

For, let m lbs. moving with u velos have mv pulses applied to it in t seconds; then, assuming that the v velos are produced uniformly, $s = (u + \frac{1}{2}v) \times t$; so that the smaller t, the smaller is s.

68. When a question on impact does not contain sufficient data to determine the magnitude of the interval and the average pressure during the impact, it is understood that *the interval is so small* that the effect of all ordinary external forces for this interval, is so much smaller than that of the pressure of the impact, that the effect of these external forces may be neglected.

This is all that is meant when it is said that '*the force of an impact is infinite compared with ordinary forces.*'

69. When a mass has received a finite addition to its velocity we may say either (i) that it has received an impulse; or (which is the same thing) (ii) that it has received the continuous action of a force for a finite interval.

That which in ordinary language we are accustomed to call a **blow** of a bat, or a **stroke** of an oar, or a **tread** of a tricyclist's foot on a pedal, each is an impulse. They each consist of a stress lasting for an interval.

One component force of the stress acting on the ball, or on the blade of the oar, or on the pedal of the tricycle, for an *interval* produces a certain number of pound-velos, in the ball, or in the boat, or in the tricycle.

It is well known both to cyclists and to oarsmen that the *longer the interval* of the impulse (the force being the same) the greater the mass-velocity produced.

Hence the importance of the *length* of the stroke in rowing.

EXAMPLES. XVI.

The following impacts are all direct.

1. An inelastic ball of 1 cwt. moving with a velocity of 20 ft. per sec. impinges against an equal ball at rest; find their subsequent velocity.

2. An inelastic particle of m lbs. having v velos impinges directly on another of m' lbs. at rest; find their subsequent velocity.

3. An inelastic particle of 20 lbs. meets another of 2 lbs.; each particle has 11 velos; find their subsequent motion.

4. A railway carriage of 10 tons is moving towards a train of 100 tons with a velocity of 4 miles an hour; at the instant of collision the carriage is coupled to the train; find the subsequent velocity.

5. A particle of 1 cwt. meets another of 28 lbs. having a velocity of 15 miles an hour and after the impact they both are at rest; with what velocity was the 1 cwt. moving?

6. A particle of 1 lb. moving with 30 velos overtakes another particle of 2 lbs. and after impact they move together with 25 velos; what was the velocity of the 2 lbs. before impact?

7. An inelastic mass impinges on another of twice its mass at rest; shew that the impinging body loses two-thirds of its velocity by the impact.

8. A golf ball is moving directly after it is struck with 500 velos assuming that the impact lasted one ten thousandth part of a second and that the golf ball is 4 ounces, what is the force of the blow?

*ELASTICITY.

70. DEF. Two masses are said to be **elastic** when, after impact, their velocities in the direction of the impulse are not the same.

When two *elastic* masses impinge they probably behave in much the same way as inelastic masses up to the instant at which the two masses are moving with the same velocity; the property of elasticity seems to cause a continuation of the impulse beyond that instant so that the masses *recoil* from each other. The amount of recoil must in some way depend on the magnitude of the impulse and on the nature of the material of the masses.

71. Newton observed, that when two spheres of given substance impinge directly on each other, their *relative* velocity after impact is e times their relative velocity before impact and in the opposite direction; where e is a number not greater than 1. This number e depends on the *nature* of the substances of the two spheres, and is called their **coefficient of restitution**; hence we say that

72. When two elastic masses m lbs. and m' lbs. impinge directly on each other, u velos and u' velos being their velocities before impact, v velos and v' velos their velocities after impact, then

$$v - v' = -e(u - u');$$

where e is a number, not greater than 1, which depends on the materials of the masses and is called the *coefficient of restitution.*

In other words, the two masses separate after impact with a velocity which is e times that velocity with which they were approaching each other before impact.

Velocity of separation $= e$ times velocity of approach.

When the coefficient of restitution of a substance $= 1$ the substance is said to be **perfectly elastic.**

Example. Two masses of 5 lbs. *and* 20 lbs. *having velocities of* 20 velos *and* -10 velos *respectively, and whose* coefficient of restitution *is* $\frac{2}{3}$, *impinge; find their subsequent velocities.*

Let v velos and v' velos be their velocities respectively after impact.

Then, since their total mass-velocity is unaltered by the impact

$$5 \times 20 - 20 \times 10 = 5v + 20v' \qquad \text{(i)}$$

also, by Art. 72,

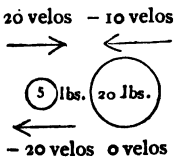
$$v - v' = -\tfrac{2}{3}(20 + 10) \qquad \text{(ii)}$$

whence $5v + 20v' = -100$,

and $v - v' = -20$;

$\therefore 25v = -100 - 400$; or, $v = -20$; also, $v' = 0$.

Hence, after the impact the 5 lbs. moves in the negative direction with velocity 20 ft. per second, and the 20 lbs. is at rest.



73. When an elastic ball impinges against any surface fixed to the Earth, its velocity after impact is $-e$ times that before impact.

For the mass of the earth is so great, that its change of velocity must be practically zero; hence the earth may be considered at rest both before and after the impact.

Suppose we have a mass of m lbs. with velocity v velos impinging on a very large mass (the Earth) M lbs., at rest.

The equations are $mu = mv + Mv'$ (i),

$v - v' = -eu$. (ii).

Now M is *very* great and therefore v' must be *very* small, so small as to be imperceptible; therefore the only equation necessary is

$$v = -eu.$$

This result gives us a rough means of testing the truth of Art. 71.

Suppose we take a glass ball and let it fall from a given height h feet on a stone floor. It reaches the floor with velocity $\sqrt{2gh}$ velos; it leaves the floor with velocity $e\sqrt{2gh}$ velos and rises to the height e^2h feet. The height to which it rises can be observed with considerable accuracy.

Now if we double the height of the fall the height of the rebound ought (neglecting the resistance of the air) to be doubled. This will be found to be approximately true.

EXAMPLES. XVII.

The coefficient of restitution is here denoted by e.
The coefficient of restitution of an elastic ball is sometimes called its elasticity.

1. A sphere of 6 lbs. having 20 velos overtakes another of 4 lbs. having 12 velos; determine their velocities after impact, in the case in which $e = \frac{1}{2}$.

2. A particle of 28 lbs. having 8 velos impinges directly on another of 14 lbs. having -16 velos; determine their subsequent velocities when $e = \frac{1}{3}$.

3. Two elastic balls ($e = \frac{1}{4}$) of masses $3m$ and m and velocities $3v$ and $-5v$ impinge; find their subsequent velocities.

4. A ball of $3m$ impinges on another m at rest which afterwards moves with 10 velos, e being $\frac{3}{5}$; find the velocity of the first ball before impact.

5. The result of a direct impact between two spheres of elasticity $\frac{1}{4}$, one of which is at rest, is that one of them has after impact twice the velocity of the other; prove that one ball has $\frac{1}{9}$ the mass of the other.

6. A series of equal elastic balls of elasticity e are placed in a line separated from each other by short distances; another ball of equal mass moving in the same line impinges on the first with velocity v; find the velocities after impact of the first, second, third and nth balls.

7. A ball of mass m and elasticity e is projected vertically under the action of gravity with 64 velos; another ball of mass m' is simultaneously let fall from a height of 64 ft. vertically above m; find when and where they impinge, and their velocities after the impact.

8. Two elastic spheres impinge directly with equal velocities; find the ratio of their masses that one of them may be reduced to rest by the impact.

9. A particle let fall from a height of 16 feet, impinges on a horizontal pavement; find the interval occupied by the first rebound, the coefficient of restitution being $\frac{1}{2}$.

10. A particle let fall from a height of 48 ft., impinges on a horizontal pavement; find the elasticity that in its first rebound it may rise to a height of 32 ft.

NEWTON'S LAWS OF MOTION.

74. We may sum up the substance of the preceding Chapters thus:—The elementary ideas considered in Dynamics are

Time, Displacement, Mass.

From these we derive

velocity which is *time-rate of displacement*,

acceleration which is *time-rate of additional velocity*,

force which produces *acceleration in mass*,

impulse which is *continuous force for an interval*, and produces *velocity in mass*,

stress which is *two equal and opposite forces*.

75. Distance, time, and mass are essentially different, and therefore velocity, acceleration, impulse, force are essentially different.

We may notice the following **dimensions**:

velocity *is of the nature* $\frac{\text{distance}}{\text{interval}}$,

acceleration „ „ $\frac{\text{velocity}}{\text{interval}} \equiv \frac{\text{distance}}{(\text{interval})^2}$,

force *is of the nature* acceleration × mass

$$\equiv \frac{\text{distance} \times \text{mass}}{(\text{interval})^2},$$

impulse „ „ force × interval

$$\equiv \frac{\text{distance} \times \text{mass}}{\text{interval}}.$$

76. Newton's Laws of Motion are

LEX I. *Corpus omne perseverare in statu quo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.*

LEX II. *Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.*

LEX III. *Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*

They may be translated as follows:

I. *Every body perseveres in its state of rest or of moving uniformly in a straight line, except in so far as it is made to change that state by external forces.*

II. *The change of the mass-velocity of a body is numerically equal to the impulse which produces it and is in the same direction.*

III. *To every action there corresponds an equal and opposite reaction; that is to say, the actions of two bodies upon each other are always equal and in opposite directions.*

With regard to II. we may notice that if the change of a quantity is always numerically equal to another quantity then the time-*rate* of change of the first quantity is numerically equal to the time-*rate* of change of the second quantity; and conversely.

Hence it follows from II. that the mass-acceleration of a body is numerically equal to the *force* which produces it and is in the same direction.

For acceleration is time-rate of change of velocity and force is time-rate of impulse.

*CHAPTER V.

CHANGE OF UNITS.

77. Our units have been, *a foot, a second, a pound;* from which are derived the units, *a velo, a celo, a poundal, a pulse.*

In all systems of units the velocity chosen for the *unit velocity* is that of a point which passes uniformly over the unit distance in the unit interval.

Similarly, the acceleration chosen for the *unit acceleration* is that of a point whose velocity is increased uniformly by the unit velocity per unit interval.

Also, the force chosen for the *unit force* is that which when acting on the unit mass produces in it unit acceleration.

Example 1. *When the* unit acceleration *is* 32 feet per sec. per sec. *and the* unit velocity *is* 10 feet per sec., *find the* unit distance *and the* unit interval.

Let the unit interval be x seconds; let the unit distance be y feet; then, by Art. 74, the unit velocity is y ft. per x seconds;

or, the unit velocity is $\frac{y}{x}$ (ft. per sec.); that is, $\frac{y}{x}$ velos. But the unit velocity is 10 ft. per second; hence, $\frac{y}{x}=10$.

The unit acceleration is (unit velocity per unit interval),
that is, 10 velos per x seconds;

or, the unit acceleration is $\frac{10}{x}$ (velos per second); that is, $\frac{10}{x}$ celos. But the unit acceleration is 32 celos; hence, $32=\frac{10}{x}$; or, $x=\frac{5}{16}$.

Also, $y=10x=\frac{50}{16}=3\frac{1}{2}$.

Thus the unit distance is $3\frac{1}{2}$ ft. the unit interval $\frac{5}{16}$ secs.

Example ii. *When the area of a field of* 10 acres *is the* unit area, *and the* acceleration due to gravity *the* unit acceleration, *find the* unit interval.

A field of 10 acres contains 48400 sq. yds., that is, $(220)^2$ sq. yds. Now the unit area is always the square described on the unit distance;

hence, the unit distance is 220 yds., or 660 ft.

Let the unit interval be x seconds.

By Art. 77, the unit velocity is 660 ft. per x seconds; that is, $\frac{660}{x}$ velos. $\therefore$ by Art. 77, the unit acceleration is $\frac{660}{x}$ velos per x seconds; that is, the unit acceleration is $\frac{660}{x^2}$ celos.

But the unit acceleration is g celos, or 32 celos;

therefore, $$32 = \frac{660}{x^2},$$

or, $x^2 = \frac{1650}{8} = 20{\cdot}625$, whence, $x = 4{\cdot}54\ldots$

The required unit is about $4\frac{1}{2}$ secs.

THE METRIC SYSTEM OF UNITS.

78. The system of units used in this book is called the Foot-Second-Pound system. These units are used by English speaking Engineers and Mechanics all over the world.

Another system is used universally in scientific investigations called the **C. G. S.** [the Centimetre-Gramme Second] system or the **Metric** system.

In this system the units are as follows:

The unit *length* is the **centimetre** = ·0328... feet.

The **metre** = 3·28... feet and the centimetre is the $\frac{1}{100}$th of a metre.

A foot = 30·48... centimetres.

The unit interval of *time* is the **second** $= \frac{1}{24} \times \frac{1}{60} \times \frac{1}{60}$ of a **day.**

A day is the period occupied by the revolution of the earth round its own axis.

The metric system was introduced by the French at the Revolution when they introduced the decimal system. They succeeded in decimalizing every system of measurement except the measures of Time and of Angle.

The unit *mass* is the **gramme**, (which was defined to be the mass of a cubic centimetre of pure water) = ·0022046 ...lbs.

A **kilogram** = 1000 grammes = 2·2046...lbs.
A **tonneau** = 1000 kilograms = 2204·6...lbs.

The unit force is the **dyne**, and is the force which produces the acceleration of a centimetre per sec. per sec. in a gramme.

We shall call the C. G. S. units of velocity and acceleration a **vélo** and a **célo** respectively.

Thus a vélo = a centimetre per second = $\frac{\text{·0328...ft.}}{\text{1 sec.}}$ = ·0328...velos

a célo = a vélo per sec. = ·0328...celos.

a dyne = a gramme × a célo
= ·00220...lbs. × ·0328...celos = ·0000722...poundals.

The acceleration due to gravity in London

$$= 32\cdot19\ldots\text{celos} = \frac{32\cdot19\ldots}{\cdot0328\ldots} \times \cdot0328 \text{ celos} = \frac{32\cdot19\ldots}{\cdot0328\ldots} \text{ célos}$$

which = ·981...célos.

The weight of a gramme (produces ·981...celos in 1 gramme) = ·981 ...dynes.

EXAMPLES. XVIII.

1. The unit distance being a yard and the unit interval a minute, express the unit velocity in velos and the unit acceleration in celos.

2. The unit distance being a mile and the unit interval an hour, express the unit velocity in velos and the unit acceleration in celos.

3. The unit distance being a yds. and the unit interval b seconds, express the unit velocity in velos and the unit acceleration in celos.

4. The unit velocity being 60 miles an hour, and the unit distance 1 yd., find the unit interval.

5. The unit velocity being 100 yds. per minute and the unit interval 5 secs., find the unit distance.

6. The unit acceleration being the double of that due to gravity and the unit distance 1 mile, find the unit velocity.

7. The unit velocity being 40 ft. in 11 secs. and the unit acceleration 3 yds. per minute per minute, find the units of time and distance.

8. The unit mass being 1 cwt. and the unit force the weight of 7 lbs., the unit velocity 10 velos, find the unit distance.

9. The unit force being the weight of 1 lb., and the unit interval and unit distance being 1 sec. and 1 ft. respectively, find the unit mass.

10. The unit mass being 1 ton, the units of interval and of distance an hour and a mile, find the unit force.

11. Taking the centimetre ($\frac{1}{100}$ of a metre = ·0328...ft.) as the unit distance, the second as the unit interval and the gram (= ·0022...lbs.) as the unit mass, find the unit velocity in velos, the unit acceleration in celos, and the unit force in poundals.

12. Prove the following statements.

The weight of a gramme in London is about 981 dynes.

A poundal = $1{\cdot}3825... \times 10^4$ dynes.

A foot-poundal = $4{\cdot}214... \times 10^5$ centimetre-dynes (ergs).

A foot-pound weight = $1{\cdot}356... \times 10^7$ ergs.

A foot-pound = $1{\cdot}3825... \times 10^4$ gramme-centimetres.

An erg (a centimetre dyne) = $7{\cdot}37... \times 10^{-8}$ foot-pound-wts.

13. Find the weight of a pound and of a gramme in poundals and in dynes in Paris ($g = 32{\cdot}18...$celos), in New York ($g = 32{\cdot}16...$celos), at the Equator ($g = 32{\cdot}09...$celos), at the Pole ($g = 32{\cdot}25$ celos).

14. If we take as our unit force the weight of 1 lb., what is the unit mass at the Equator, in Paris, in London, in New York? [See **13.**]

SECTION II.

MOTION IN ONE PLANE.

CHAPTER VI.

DIRECTION.

79. Displacement has **direction.**

We have hitherto supposed all our displacements to be measured in the same direction, (or in exactly opposite directions), so that it was only necessary to consider the *magnitude* and *sign* of our displacements.

In what follows we shall always consider the *direction* of a displacement to be an essential part of it. As a necessary consequence we must consider *direction* to be an essential part also of all quantities which vary directly as displacement, such as *velocity*, *acceleration*, *force*, *impulse*.

It is convenient to use the word **speed** to denote the *magnitude* only of a velocity; just as we use the word *length* to denote the *magnitude* only of a displacement. The **average speed** of a point moving in a *curved* line is that which varies directly as the *length* of that portion of the curved line passed over by the point, and inversely as the interval occupied.

The average *speed* of a point moving in a *straight* line is the same as its average *velocity*.

When we speak of a train moving at the rate of 40 miles an hour for a certain interval, we mean that its *average speed* is 40 miles per hour.

A point may have a constant speed while its velocity is changing (in direction).

80. PROP. Any quantity which has direction and magnitude may be represented in a diagram by a finite straight line with an arrow-head.

For the direction of the quantity can be represented by the *direction* of the line with the arrow-head, and the numerical measure of the quantity can be indicated by that of the *length* of the line.

When a line in a diagram is referred to by the two letters at its extremities, the direction of the arrow-head may be indicated by *the order of the letters*.

81. A displacement can always be found which represents the *combined effect*, or the *resultant, of two given displacements in different directions.*

The combined effect of, (or the result of adding together) two non-directional, (or scalar) quantities is the arithmetical sum of their measures.

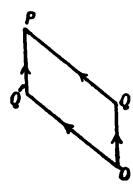
The same is true of two directional, (or vector) quantities which have the same direction.

But the combined effect of two directional, (or vector) quantities having different directions requires definition.

Example. A point P is said to have a displacement of 2 inches northwards from a given point O, and *also* a displacement of 3 in. towards the north-west from O, when the position of P is found as follows:

Starting from O measure OQ 2 in. towards the north; then *from* Q measure QP 3 in. towards the north-west.

Or thus; starting from O measure OQ' 3 in. towards the north-west and then *from* Q' measure $Q'P$ 2 in. towards the north.



It will be seen that the resulting position of P is the same in each case; for OQ, $Q'P$ are equal and parallel and therefore also QP is equal and parallel to OQ'. Euc. I. 33.

82. We may illustrate the meaning of a point having two **simultaneous displacements** thus:

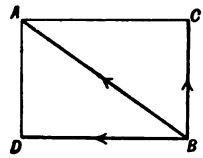
Suppose a sheet of glass to be placed on paper and the point P to be a moveable mark placed on the *glass*.

Let the mark be moved a distance of 3 in. on the glass towards the north-west, and let the glass be moved relatively to the paper, a distance of 2 in. northwards.

It will be seen that the resulting position of the point P with reference to the paper is that found as in the Example in Art. 81.

And it makes no difference to the ultimate position of P whether these displacements are made *simultaneously*, or whether they are made *separately*, one of the movements being finished before the other is made.

Example i. *A point* A *is distant* 3 miles *towards the* North *and* 4 miles *towards the* West *from a point* B; *what is the* total distance *of* A *from* B?



Draw BC northwards 3 units of length; draw BD westwards 4 units of length; complete the parallelogram $CBDA$;

Then BA represents the required total distance.

By construction, $CBDA$ is a right-angled parallelogram,

therefore $AB^2=AD^2+BD^2$. [Euc. I. 47.]

Let AB contain x units of length; then, by the above statement,

$$x^2=3^2+4^2=9+16=25;$$

therefore $x=5$.

Thus the distance AB contains 5 units of length.

Each unit in the above diagram represents a mile.

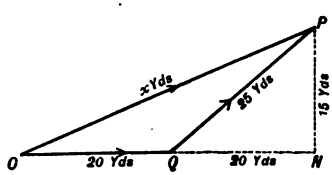
Hence it follows that the total distance of A from B is 5 miles.

And the direction is such that if AB make an angle α with BD,

then, $$\tan\alpha=\frac{DA}{BD}=\frac{3}{4}.$$

Example ii. *A man walks* 20 yards eastwards *and then* 25 yards *in a direction making an* angle whose sine is $\frac{3}{5}$ *with the line pointing eastwards and on the north side of it; find the* direction and magnitude of the resultant displacement *of the man from his starting point.*

Let OQ represent the displacement of 20 yards eastwards; produce OQ to N; let QP represent the displacement of 25 yds., QP being drawn so that $\sin PQN=\frac{3}{5}$.



Draw PN perpendicular to OQ produced; then,

$\frac{NP}{QP}$ is the sine of the angle which QP makes with ON; therefore $\frac{NP}{QP}=\frac{3}{5}$.

Let PN contain y yds. Then $y = \frac{3}{5}$ of $25 = 15$.

Similarly, QN may be shewn to contain 20 yds.

Let OP contain x yds. Then since $OP^2 = ON^2 + NP^2$, [Euc. I. 47] therefore, $x^2 = (20+20)^2 + (15)^2 = 1600 + 225$

$= 1825$;

therefore, $x = \sqrt{(1825)} = 42{\cdot}7\ldots$

Hence, the required displacement is 42·7 yds. Also

$$\tan PON = \frac{NP}{ON} = \frac{15}{40};$$

therefore its direction makes an angle whose tangent is $\frac{15}{40}$, or $\frac{3}{8}$, with the line towards the East.

N.B. The student should also solve this question with the scale and protractor. That is, he should draw a figure very carefully to scale of OQ and QP and then join OP and by measuring OP find what length it represents.

EXAMPLES. XIX.

1. A man walks 300 yards northwards and then 400 yards westwards; another man starting from the same place walks 400 yards westwards and then 300 yards northwards; shew that the two men are then at the same total distance from the starting point; and find that distance.

2. A dog on an ice-floe goes 3 miles northwards while the floe itself travels 4 miles eastwards; how far is the dog then from the place on the earth's surface from which he started?

3. A fly crawls along the floor of a railway carriage a distance of 2 ft., at right angles to the line of rails, and at the same time the carriage moves a distance of 10 ft.; find approximately the distance moved over by the fly.

4. A man swims across a river, and in 40 seconds he swims 20 yards; in the same time the current of the river has carried him 15 yards down stream; how far is he from his starting point after 40 secs.?

5. A fly, in a railway carriage which is moving at the rate of 10 ft. per sec. northwards, crawls at the rate of 5 ft. in 10 seconds across the floor of the carriage towards the north-east; what distance will the fly have gone in 10 seconds?

6. On the floor of a railway carriage which is moving with 4 velos northwards, four flies A, B, C, D start from the same place and crawl each with 3 velos, A Northwards, B Southwards, C Westwards, and D towards the North-West. Find their displacements from the starting point after 2 seconds.

* *The following examples require the use of the Trigonometric Ratios, or of a scale and protractor as in Example* ii. *p.* 64.

7. A man walks 350 yards in a certain direction and then turning his direction through an angle of 120 degrees walks 350 yards more; find the direction and magnitude of his resulting displacement from his starting point.

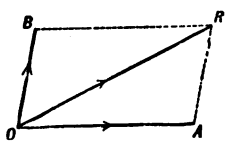
8. A man on the deck of a ship moving due north, walks 20 yards towards the north-east while the ship goes 100 yds.; find the direction and magnitude of the whole distance passed over by the man.

9. The distance of a point *B* from another *A* is the sum (i.e. the resultant) of two equal distances of 20 yds. inclined to each other at an angle of 60 degrees; find the distance of *B* from *A*.

10. When the wind is due North a certain boat can sail either towards the north-east or north-west; it sails first 2 miles towards the N.E., then 2 miles towards the N.W., then 1 mile more towards the N.E.; find the magnitude and direction of its resulting displacement from its starting point.

83. DEF. The **resultant** of two *displacements* is the one *displacement* which is the combined effect or sum of the two displacements.

84. PROP. **The Parallelogram of Displacements.** *The resultant of two given displacements* OA, OB *is* OR, *the diagonal of the parallelogram* AOB.



This follows because the opposite sides of a parallelogram are equal.

In the figure below, *OP* is the resultant of the displacements *OM* and *MP*; that is, of *OM* and *ON*, for *MP* is equal and parallel to *ON*.

85. When the two displacements are in the same direction their resultant, found by the *parallelogram law*, coincides with the two displacements placed end and end; so that its magnitude is their *sum*.

When the two displacements are in opposite directions the magnitude of their resultant is their *difference*.

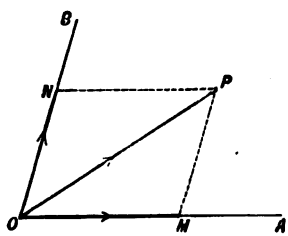
86. It is often convenient to express a displacement as the resultant of two displacements in given directions.

Example. Express the displacement OP *as the resultant of two displacements in directions parallel to* OA *and* OB *respectively.*

Through *P* draw *PM* and *PN* parallel to *OB* and *OA* respectively.

Then *OP* is the resultant of the displacements *OM* and *ON*.

87. DEF. When the one displacement *OP* is expressed as the sum of two others *OM* and *ON*, the two displacements *OM* and *ON* are called the **components** of *OP*; and *OP* is said to be **resolved** into its two *components OM* and *ON* in those directions.



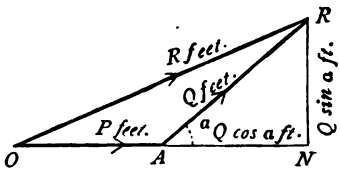
The distance *OP* is clearly the *resultant* of its two components *OM* and *ON*.

It is usually convenient to take the two component directions **at right angles** to each other.

DEF. When the two components are mutually at right angles, each component is called the **resolute** of the resultant in its own direction.

88. The most convenient method of **calculating** the resultant of a number of distances (or other directional quantities) is by taking their resolved parts in two directions mutually at right angles.

Example. A man walks P feet *in a straight line a then turning through the angle a walks* Q feet *straight on; shew that he is then at the distance* $\sqrt{(P^2+Q^2+2PQ\cos a)}$ feet *from the starting point.*



Let OA represent P feet, AR represent Q feet; let OR be R feet. Then $R^2=P^2+Q^2+2PQ\cos a$.

For if we draw RN perpendicular to OA we have

$$AN=AR\cos a=Q\cos a \text{ ft.}; \quad NR=AR\sin a=Q\sin a \text{ ft.}$$

and $$OR^2=ON^2+NR^2=(OA+AN)^2+NR^2$$

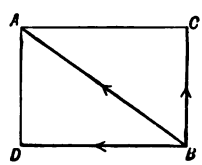
$$\therefore R^2=(P+Q\cos a)^2+Q^2\sin^2 a$$

$$=P^2+2P.Q\cos a+Q^2(\cos^2 a+\sin^2 a)$$

$$=P^2+2P.Q\cos a+Q^2. \qquad \text{Q. E. D.}$$

The most convenient method of obtaining the resultant **practically** is to draw a diagram to scale and measure the resultant with a rule.

Example i. *Resolve a given displacement of* 1000 yards *into two displacements mutually at right angles, one of which makes the* angle whose cosine is $\frac{4}{5}$ *with the given displacement.*



Let BA represent the given displacement of 1000 yards.

Let ABD represent the angle whose cosine is $\frac{4}{5}$.

Draw BC perpendicular to BD and complete the rectangle $ADBC$.

Then by construction $\dfrac{DB}{AB}=\frac{4}{5}$, $\therefore DB=\frac{4}{5}$ of AB

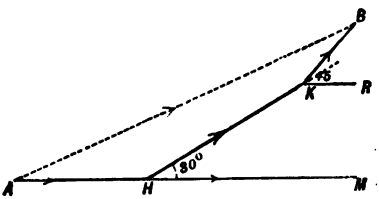
$$\therefore DB=\tfrac{4}{5} \text{ of } 1000 \text{ yards} = 800 \text{ yards}$$

and $$AD^2=AB^2-BD^2=(1000)^2-(800)^2=360000=(600)^2.$$

Thus the two required displacements are 800 yards and 600 yards respectively.

Example ii. *The road from* A *to* B *runs in a straight line for* 2 miles *from* A *to* H; *then after turning through an angle of* 30°, *it runs in a straight line for* 3 miles *from* H *to* K; *then after turning through an angle of* 15°, *runs in a straight line for* 1 mile *from* K *to* B; *how far is it from* A *to* B *as the crow flies?*

Take a scale and draw on paper a line AM; mark off AH say two inches long; make the angle $MHK=30°$; measure off $HK=3$ inches; draw KR parallel to AH; make the angle $RKB=45°$; measure off $KB=1$ inch; then AB is the required distance; on measuring AB it should be found that $AB=5\frac{3}{4}$ inches; therefore the distance from A to B direct is about $5\frac{3}{4}$ miles.



EXAMPLES. XX.

1. Resolve a distance of 100 yards into two components mutually at right angles one of which makes an angle of 30° with it.

2. Resolve a distance of 25 yards into two components mutually at right angles one of which makes an angle whose sine is $\frac{4}{5}$ with it.

3. A man walks 200 yards in a straight line, then turns through an angle of 45° and walks 150 yards further in a straight line; find his resultant distance from the starting point.

4. Find the resultant of two distances of 300 yards and 450 yards making an angle of 45° with each other.

5. I walk 1 mile towards the north-east; what is the *resolved part* of this distance in the northerly direction? what is the *resolved part* of this distance towards the north-west?

6. Find the resolved part of a distance of 360 yards in the direction which makes an angle of 60° with it.

7. A man walks a mile straight; he then turns through 27° and walks 2 miles straight; what is his total displacement? [cos 27°=·883].

8. A man walks 2 miles straight; he then turns through 54° and walks 5 miles straight; how far is he now from the starting point? [cos 54°=·588....]

89. DEF. When a point has a given uniform velocity u velos in a given direction, then in the interval t seconds the point is displaced ut feet in the direction of the velocity. This displacement is called **the displacement due to the velocity.**

90. DEF. A point is said to have **simultaneous** velocities and accelerations when its displacement for any interval is the resultant of the displacements due to those velocities and accelerations separately.

Example. A point has -32 celos *vertically upwards, and at a certain instant it has* 200 velos *vertically upwards, and* 40 velos *in a horizontal direction; find its position after* 3 seconds.

Let O be its initial position.

Draw ON horizontally to represent 3×40 ft.

From N draw NQ vertically upwards to represent 3×200 ft.
and from Q draw QP vertically downwards to represent $\frac{1}{2} \times 32 \times 9$ ft.

Then P represents the position of the point after 3 seconds.

EXAMPLES. XXI.

Find the resultant displacement of a point having the following;

1. 3 velos northward and 4 velos eastwards, after 30 seconds.
2. 2 velos southwards and 3 velos eastwards, after 15 secs.
3. 4 velos northwards and 7 velos southwards, after 8 secs.
4. 20 velos northwards, 15 velos eastwards and 15 velos northwards, after 8 secs.
5. 20 velos northwards and -3 celos westwards, after 10 seconds.
6. Three velocities of 20 velos each making an angle of 120° with each other, after 10 seconds.
7. 32 celos westwards, 20 velos northwards, initially after 2 secs.
8. Three accelerations, 15 celos westwards, 20 celos northwards and 15 celos eastwards after 1 second, initially at rest.

91. A particle which having been projected from any point in any direction then moves under the action only of its own weight is called a **projectile.**

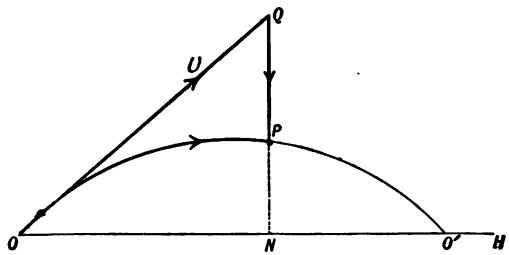
The motion of a **projectile** affords an important example of a point which after any interval has two simultaneous displacements.

Example i. *A particle is projected with* u velos *in a direction making an* angle α *with the horizontal plane; find its position after* t seconds (*neglecting the resistance of the air*).

It is understood that the only force acting on the particle is its own weight, which acts vertically downwards.

By Art. 33 this force produces in the mass the same acceleration whether the particle has other velocity or not.

Hence, the *displacement* of the particle for t seconds is the resultant of two displacements, one due to the u velos, the other due to the acceleration g celos vertically downwards.



Let O be the initial position of the particle P; let OH be a horizontal line; then we may find the position of P after t second thus:

Draw OQ, making an angle α with OH, so that OQ represents a distance of ut feet;

from Q draw QP vertically downwards, so that QP represents $\frac{1}{2}gt^2$ feet.

Then P is the position of the particle after t seconds.

For OQ is the displacement due to the velocity u velos, and QP is the displacement due to the acceleration g celos†.

Example ii. *A stone is thrown from a point* O, *which is distant* 100 yds. *from* Q *the top of a tower, with the velocity* 100 velos *in the direction* OQ; *find where the stone will strike the tower.*

In the above figure let QN be the tower, O the point of projection; then $OQ = 100$ yds. $= 300$ feet.

If gravity were not acting the stone would go straight from O to Q with 100 velos in 3 seconds.

Since gravity acts, the stone describes the curved path from O to P; at the end of 3 seconds its position is found by drawing $OQ = 300$ ft. and then QP vertically downwards so that $QP = \frac{1}{2} \times 32 \times 3^2$ feet $= 144$ feet.

Thus the stone will strike the tower 144 feet below the point Q.

† The student is advised to find the position of the projectile at short intervals, by drawing the lines representing its distances carefully to scale; he will in that way trace out the *path* of the projectile.

EXAMPLES. XXII.

Find the positions of the three following projectiles after intervals of 1, 2 and 3 seconds respectively from the instant of projection;

1. of a stone thrown from the top of a cliff in a horizontal direction with 100 velos.

2. of a stone thrown at an angle of 45 degrees to the horizon with 100 velos.

3. of a bullet projected at an angle of 30° to the horizon with 1000 velos.

4. A rifle is pointed at the top of a tower 100 ft. high standing on a horizontal plane, the top being at a distance of 800 yards from the rifle. What must be the velocity of projection that the bullet may strike the foot of the tower?

5. A rifle is pointed at the top of a tower which is distant 1000 yards from the rifle; the bullet is projected with 800 velos; where will it strike the tower?

6. A vessel sailing southwards at the rate of $7\frac{1}{2}$ miles an hour points a gun horizontally at a vertical rock whose face, distant $\frac{1}{4}$ mile from the ship, is parallel to the path of the vessel; if the muzzle velocity of the shot is 1320 velos, how many feet from the spot at which the gun was pointed is the spot at which the shot will strike the rock?

92. PROP. **The path of a projectile** (*or of any moving point which has a* constant acceleration in a fixed direction) **is a parabola.** [Compare p. 46.]

Let P be the position of the point at any chosen initial instant.

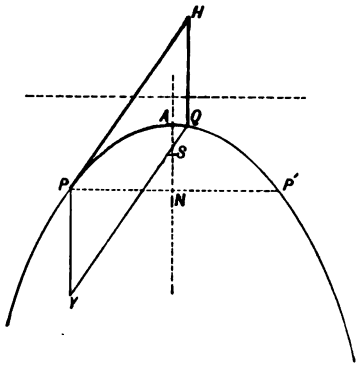
Let PV be the direction of its constant acceleration a celos.

Let PQ be the direction of its velocity u velos at the initial instant.

Then the moving point has simultaneously u velos along PH and a celos along PV.

The displacement PH of the moving point due to the u velos is such that $PH = ut$ feet.

The displacement PV of the moving point due to the a celos is such that $PV = \frac{1}{2}at^2$ feet.



The actual position Q of the moving point after t seconds is found by completing the parallelogram $PVQH$.

Now $QV^2 = HP^2 = u^2t^2$ and $PV = \frac{1}{2}at^2$.

Whence $QV^2 = \frac{2u^2}{a} PV$.

Therefore Q always lies on the *parabola* in which $QV^2 = 4SP \,.\, PV$, where $4SP = \frac{2u^2}{a}$. Q.E.D.

93. PROP. *The* **velocity** *of a projectile at any point is that due to a fall from the directrix.*

The distance of the point P in the last article from the directrix is SP and $SP = \frac{u^2}{2a}$

or $u^2 = 2a \,.\, SP$

$= 2g \,.\, SP$ in the case of a projectile.

That is the velocity u at P is that gained while moving over the distance SP under the acceleration g.

Since a projectile may be considered to be projected from any point in its path, the proposition is true.

CHAPTER VII.

THE PARALLELOGRAM OF VELOCITIES.

94. DEF. The **resultant** of two simultaneous *velocities* is a *velocity* such that the displacement due to it for any interval is equal to the resultant of the two displacements due to the two simultaneous velocities.

95. The two simultaneous velocities are called the *components* of the resultant velocity; and when they are mutually *at right angles* they are each called the *resolute* of the resultant in their own direction.

96. In fact it will be found that all definitions and propositions applicable to displacements may be applied to velocities (and also to any *directional* quantity). Thus we have

97. PROP. **The Parallelogram of Velocities.** *When two velocities are represented (in direction and magnitude) by two straight lines* OA *and* OB, *their resultant is represented (in direction and magnitude) by* OR *the diagonal of the parallelogram* AOB.

Let a point P have (in addition to any other velocity) the two simultaneous velocities represented by OA and OB.

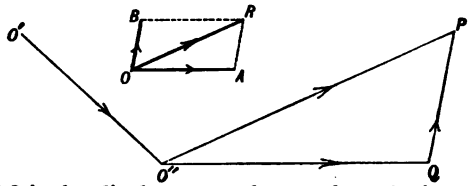
We have to prove that OR represents the resultant of the two simultaneous velocities represented by OA, OB.

Let OA represent u' velos; let OB represent u'' velos; and let OR represent u velos.

Let O' represent the position of the moving point at the beginning of an interval t seconds.

Let $O'O''$ represent the displacement due to all the other velocities of the moving point for the interval t seconds.

Draw $O''Q$ parallel to OA to represent $u't$ ft.; draw QP parallel to OB to represent $u''t$ ft. Then P is the actual position of the moving point at the end of the interval.



$O''Q$ is the displacement due to the velocity OA,

QP is the displacement due to the velocity OB,

and therefore $O''P$ is the displacement which must be due to the velocity which is the resultant of OA and OB.

But, $$\frac{O''Q}{O''P} = \frac{OA}{OR} = \frac{u'}{u}$$

and since $O''Q = u't$ ft. $\therefore$ $O''P$ represents ut ft.

That is, $O''P$ is the displacement due to the velocity u represented by OR.

But $O''P$ is the displacement due to the resultant velocity.

Therefore OR represents the resultant of the velocities OA, OB.

98. When the moving point has only the two velocities OA and OB, then in Art. 97, $O'O''$ disappears, and we see that since $O''P$ is always parallel to OR (which is fixed in direction and magnitude) and always represents ut ft., the *actual motion* of the moving point is the velocity OR.

99. The resultant of a number of simultaneous velocities is a uniform velocity.

For any two of them may be replaced by their resultant; this resultant and a third velocity may be replaced by their resultant; and so on.

PROP. *The actual motion of a point which is moving with a number of simultaneous uniform velocities only is uniform velocity.*

For its resultant velocity, given by the parallelogram of velocities, is always the same; and its resultant displacement from its initial position is always parallel and proportional to this resultant velocity.

Its actual path is therefore a straight line along which it moves with uniform velocity.

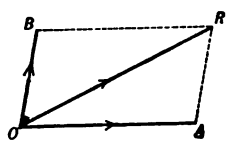
100. PROP. *When a point is moving with given variable velocities* **its actual velocity at any instant** *is the resultant of the velocities which it has at that instant.*

For, consider another point Q moving with uniform velocities such that at that instant it has the *same* simultaneous velocities as the point P, then Q will, at that instant, be moving in the same direction with the same speed as the first point P.

101. The parallelogram of Velocities is sometimes stated in the following form, which is called

The Triangle of Velocities. When a point has two simultaneous velocities, the resultant velocity may be found as follows: Draw a line OA to represent one of the velocities; from A draw a line AR to represent the other velocity; the third side OR of the **triangle** OAR represents the resultant velocity of the two simultaneous velocities OA, AR.

This proposition is only another way of stating the parallelogram of velocities.



For in the parallelogram $OARB$ it is clear that if OA, OB represent two simultaneous velocities, then AR represents a velocity equal and parallel to that represented by OB.

102. *To find the resultant of any number of velocities represented by straight lines* OA, OB, OC,...

Draw from A a line AR' parallel and equal to OB; from R' a line $R'R''$ parallel and equal to OC; and so on: the line joining O to the last point R represents the resultant of all the velocities.

Or by calculation as follows; [See Art. 88].

Resolve each velocity into its resolved parts in two convenient directions, mutually at right angles; add the velocities in each of those directions and find the resultant of the two sums.

Example i. The student is advised to use the method of Example ii. p. 64 in the following.

Find the resultant of the velocities 20 velos, 30 velos *and* 10 velos, *the angles between their direction being* 30° and 15° *respectively.*

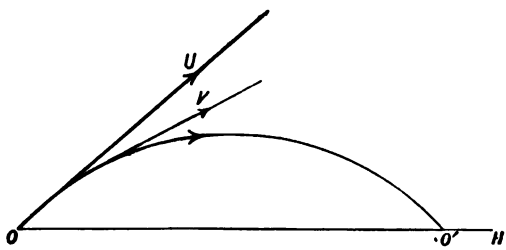
Example ii. *A stone is thrown with* 33 velos *from a train, at right angles to its direction of motion, while the train is going* 30 miles an hour ; *find the* initial velocity *of the stone relatively to the ground.*

Relatively to the ground the stone has two simultaneous velocities of 33 velos and 44 velos respectively, at right angles to each other.

Therefore, by the parallelogram of velocities, their resultant is a velocity of 55 velos making with the direction of the train an angle whose tangent is $\frac{3}{4}$.

Example iii. *A particle is projected with given velocity under the action of gravity; find its* velocity *after* t *seconds.*

Let the velocity of projection be U velos in the direction α with the horizon.



Let u be the horizontal and v the vertical resolute of U [so that $u = U\cos\alpha$, $v = U\sin\alpha$].

The particle has g celos downwards.

Therefore the horizontal resolute of the velocity V of the particle after t seconds is still u velos

and the vertical resolute of V after t seconds is $(v - gt)$ velos.

Therefore $$V^2 = u^2 + (v - gt)^2.$$

The direction of V makes the angle θ with the horizon such that $\tan\theta = \frac{v - gt}{u}$.

NOTE. When the resultant velocity of a point is zero, the point is at rest.

For the *displacement* due to all its velocities after any interval is also zero.

103. When a point at one instant has a certain velocity and at another instant has a different velocity, then some velocity must have been added to its original velocity in the interval.

The velocity which has been added may be found by the parallelogram of velocities; for the new velocity is the resultant of the original velocity and of the velocity added.

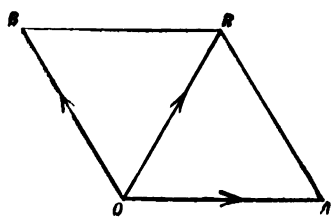
Example i. *A point in a certain interval has its velocity changed from* 20 velos *in a given direction to* 20 velos *in a direction making an* angle 60° *with the first direction; find the* additional velocity.

Let OA represent the original velocity; let OR represent the new velocity; join RA; draw OB parallel and equal to AR.

Then OB represents the additional velocity required.

The angle $AOR=60°$ and $OA=OR$, therefore AOR is an equilateral triangle; hence the angle $ROB=ORA=60°$.

Therefore $OB=OA$,



Thus, the additional velocity is a velocity of 20 velos in a direction making an angle 120° with the original velocity.

Example ii. *A ship is steaming due West with the velocity* 15 miles per hour. *The wind appears to a man on deck to be blowing from the North West, but its real direction is from the North. Shew that the velocity of the wind is* 22 velos.

The motion of the ship towards the West gives to the wind the additional velocity relative to the man of 15 miles per hour (22 velos) from the West. Thus the velocity of the wind relative to the man, which is from the N. W., is the resultant of two velocities from the N. and W. respectively. These two velocities are such that their resultant bisects the angle between them.

Therefore the velocities are equal by the parallelogram of velocities.

EXAMPLES. XXIII.

Find the resultants of the following 12 pairs of velocities:

1. 12 velos and 16 velos mutually at right angles.

2. 20 velos and 15 velos mutually at right angles.

3. 30 miles an hour and 40 miles an hour at right angles to each other.

4. 2 velos and 8 velos mutually at right angles.

5. a velos and b velos at right angles to each other.

6. a ft. per second and m miles per hour mutually at right angles.

In the following questions use a scale and protractor when necessary.

7. 3 velos and 3 velos inclined at an angle of 120°.

8. a velos and a velos inclined at an angle of 60°.

9. 3 velos and $3\sqrt{2}$ velos inclined at an angle of 45°.

10. 10 velos and 20 velos inclined at an angle of 60°.

11. 40 velos and $20\sqrt{3}$ velos inclined at an angle of 150°.

12. 20 velos and 25 velos inclined at an angle whose sine is $\frac{4}{5}$.

13. Find the displacement in 10 seconds due to 3 simultaneous velocities of 2 velos, 3 velos and 1 velo making the two angles of 30° and 15° with each other.

14. A particle is given a velocity of 50 velos in a direction making an angle 45° with the horizon; resolve this velocity into two; one horizontal and the other vertical.

15. A particle has velocity 100 velos making an angle whose sine is $\frac{1}{11}$ with the horizon; what is the resolved part of the velocity in a horizontal direction?

16. From a ship sailing at the rate of 25 miles an hour a bullet is fired with a muzzle velocity of 1100 feet per second in a direction at an angle whose sine is $\frac{1}{15}$ with the direction of the ship's motion; find the initial velocity of the shot relative to the water.

17. A football moving with 20 velos receives a kick, after which its direction is changed through a right angle but its speed is unaltered; what velocity was given it by the kick?

18. A particle travelling northwards with 20 velos, receives an additional velocity of 20 velos so that the resultant velocity is 20 velos; in what direction was the additional velocity?

19. The direction of a point's motion is altered by 45° while its speed is unchanged; what is the alteration in its velocity?

20. A cricket ball having 30 velos is struck by a bat so that its velocity is changed to 40 velos in a direction making an angle 150° with its former direction; what velocity was given to the ball by the bat?

21. A stone thrown from a train has a certain velocity given it at right angles to the path of the train, so that relatively to the ground it has 60 velos in a direction making 60° with the path of the train; find the velocity of the train.

22. A particle is projected with a velocity whose vertical and horizontal components are 50 velos and 10 velos respectively; the vertical velocity is diminished at the rate of 32 velos per second, the horizontal velocity is unaltered; find (the magnitude and direction of) the velocity of the particle after 1 second. What was the direction in which it was projected?

23. A spherical shot is rolling directly across the smooth horizontal deck of a ship with a velocity 10 velos; find where it would strike the side of the ship supposing the ship, which is going 10 miles an hour, to be suddenly stopped when the shot is 20 ft. from the side.

24. A balloon rising vertically with a velocity of 10 miles an hour is carried by the wind over a horizontal distance of 100 yds. in 20 seconds; find the velocity of the balloon.

THE PARALLELOGRAM OF ACCELERATIONS.

104. DEF. The **resultant** of two simultaneous *accelerations* is an acceleration, such that the distance due to it after any interval is the resultant of the distances due to the two accelerations.

105. **The Parallelogram of Accelerations.** *When two simultaneous accelerations are represented (in direction and magnitude) by two straight lines* OA, OB, *their resultant is represented (in direction and magnitude) by* OR, *the diagonal of the parallelogram* AOB.

Let a point have (besides any other velocities) the two simultaneous accelerations represented by OA and OB.

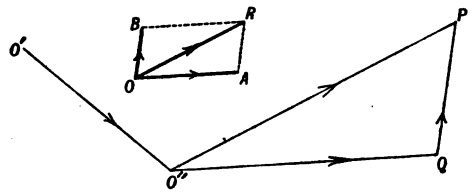
We have to prove that OR represents the resultant of the two accelerations represented by OA, OB.

Let OA represent a' celos; let OB represent a'' celos; let OR represent a celos.

Let O' represent the initial position of the moving point.

Let $O'O''$ represent the displacement due to all the other velocities of the moving point for the interval t seconds.

Draw $O''Q$ parallel to OA to represent $\frac{1}{2}a't^2$ ft., draw QP parallel to OB to represent $\frac{1}{2}a''t^2$ ft. Then P is the actual position of the moving point at the end of the interval.



Then $O''Q$ is the displacement due to the acceleration OA, QP is the displacement due to the acceleration OB, and therefore $O''P$ is the displacement due to their resultant.

But $$\frac{O''Q}{O''P} = \frac{OA}{OR} = \frac{a'}{a},$$

and since $O''Q = \frac{1}{2}a't^2$ ft., $\therefore$ $O''P = \frac{1}{2}at^2$ ft.

That is, $O''P$ is the displacement due to the acceleration a represented by OR.

But $O''P$ is the displacement due to the resultant acceleration.

Therefore OR represents the resultant of the acceleration OA, OB.

106. When a moving point is moving with given **variable** simultaneous accelerations, *its actual acceleration at any instant* is found by the parallelogram of accelerations.

For, consider another point Q moving with uniform accelerations, such that at that instant it has the *same* velocities and accelerations as the first point P; then Q will, at that instant, be moving with the same actual velocity and with the same actual acceleration as the first point P.

107. The resultant of a number of simultaneous accelerations may be found in exactly the same way as the resultant of a number of simultaneous distances or velocities. [See Art. 97.]

108. When the resultant of all the accelerations which a point has is zero, the point is moving with uniform velocity.

For the motion is that of a point having no acceleration.

109. When a point has simultaneous accelerations, each acceleration adds its own velocity to the initial velocity of the point independently of any other velocities which the point may have. Hence

110. The parallelogram of accelerations may be deduced from the parallelogram of velocities in precisely the same words as the parallelogram of velocities was deduced from that of distances by reading velocity for distance and acceleration for velocity in Art. 97.

111. The parallelogram of accelerations may also be stated as the **Triangle of Accelerations** in the words of Art. 101.

EXAMPLES. XXIV.

1. A man is starting to leave a railway carriage walking directly towards the door with an acceleration 3 celos, and the train itself is stopping with retardation 4 celos; what is the direction of the resultant acceleration of the man?

2. A man in a train which is getting up its speed with an acceleration 5 celos, gives a ball an acceleration 12 celos, in the direction at right angles to the motion of the train; find the resultant acceleration of the ball.

3. Supposing in Question 2 the man commences to throw the ball just as the train is starting and he keeps up the pressure on the ball with his hand (thus producing 12 celos) for 1 second, find the initial velocity of the ball on leaving his hand.

4. Supposing in Question 2 the train is moving at the rate of 30 miles an hour (and with 5 celos) when the man commences his throw, and that he keeps up the pressure for 1 sec., find the initial velocity of the ball.

5. A man moves across a railway carriage which is 7 ft. across with an acceleration of 5 celos; the carriage is moving with a uniform velocity of 30 miles an hour; find the velocity with which he leaves the carriage.

6. A train is pulling up with a retardation of 10 celos; I wish to walk straight across the carriage with an acceleration of 10 celos; in what direction shall I have to apply a horizontal force to my body in order to do so?

SECTION III.

FORCES ACTING AT A POINT.

CHAPTER VIII.

THE PARALLELOGRAM OF FORCES.

112. By Arts. 31, 33 every *force* applied to a mass produces in that mass its proper *acceleration* in its own direction independently of any other motion the mass may have.

It follows that every force acting upon a mass for a given interval produces in that mass its own proper velocity in its own direction; in other words that every *impulse* applied to a mass produces in that mass its proper *velocity* in its own direction, independently of any other motion the mass may have.

Illustration. Suppose a man, in a railway train moving with 44 velos westwards, to apply to a mass (a stone) a force northwards for a certain interval. He applies an impulse northwards and therefore gives to the stone a velocity northwards; say 33 velos. The stone has also the velocity of the train. The stone has therefore on leaving the man's hand two simultaneous velocities, one of 44 velos westwards and another of 33 velos northwards. Its motion relative to the earth will be a velocity of 55 velos in a direction between North and West. On leaving the man's hand the stone has at once an acceleration due to gravity of 32 celos downwards as explained in Art. 40.

113. When two forces act simultaneously on a mass they produce two simultaneous accelerations; each force producing in the mass its own proper acceleration in its own direction.

In Art. 105 it is proved that two simultaneous accelerations may be replaced by a single acceleration, hence

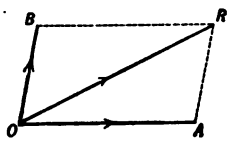
DEF. The **resultant** of two *forces* is that *force* which acting on any mass will produce in that mass the resultant of the accelerations which the two forces produce when acting simultaneously on the same mass.

114. PROP. **The Parallelogram of Forces.** *When two forces acting on a particle are represented (in direction and magnitude) by the straight lines* OA, OB, *their resultant is represented (in direction and magnitude) by* OR *the diagonal of the parallelogram* AOB.

By Art. 33, when forces act simultaneously on a particle each force produces in the particle an acceleration in its own direction proportional to the force producing it.

Hence forces which act upon the *same* particle may each be represented by that straight line which represents the acceleration which it produces in the particle.

But, by Definition, the *resultant* of two forces is that force which produces in a particle the resultant *OR* of the two accelerations *OA*, *OB* produced in the particle by the forces.



Therefore the resultant of two forces represented by two straight lines *OA*, *OB*, is *represented* by *OR* the diagonal of the parallelogram *OARB*. Q. E. D.

115. PROP. **The Parallelogram of Impulses** follows immediately upon the parallelogram of velocities.

For, since the velocities which any given impulses produce in the same mass are proportional to and in the same direction as the impulses which produce them, it follows that *impulses*, whether simultaneous or not, which act upon the same particle may be represented by the lines which represent the *velocities* they produce.

Therefore the resultant of two impulses *OA*, *OB* is represented by *OR* the diagonal of the parallelogram *OARB*.

116. In considering the motion of a mass acted on by several forces either

I. We may consider that each force produces its own acceleration in the mass, and *then* find the acceleration of the mass by finding the resultant of all these accelerations, or

II. We may first find the resultant of all the forces, and *then* consider that the forces are equivalent to this resultant only, which resultant produces the actual acceleration of the mass.

These two ways of considering the motion of a mass are identical; for since the forces act upon the *same mass*, the forces and the acceleration they produce may be represented by the same lines.

It should be noticed that the three propositions called the Parallelogram of Displacements, the Parallelogram of Velocities, and the Parallelogram of Accelerations are *geometrical facts;* while the proposition known as the Parallelogram of Forces is deduced from the Parallelogram of Accelerations, by means of the definition of Force as that which produces acceleration in mass, each force producing its own acceleration exactly as if the others did not exist.

When the resultant of any number of forces is zero, they produce no resultant acceleration; in other words, they have no effect on the motion of any mass on which they simultaneously act.

Forces whose resultant is zero are said to be in **equilibrium.**

As a particular case we may notice, that when one of three forces is equal and opposite to the resultant of the other two, the three forces are in equilibrium.

DEF. The force which is equal and opposite to the resultant of a system of forces is called

the **Anti-resultant**

of that system of forces.

117. The student should notice that the parallelogram of forces includes the case of two forces acting in the same straight line.

For suppose OA and OB be two forces including a very small angle [Fig. i.], then OC becomes very nearly equal to $OA+AC$, that is to $OA+OB$.

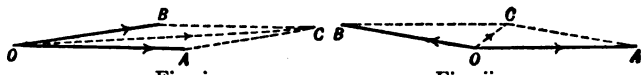
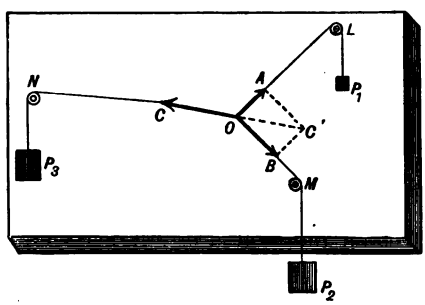


Fig. i. Fig. ii.

Again, [Fig. ii.] suppose OA and OB to include nearly two right angles, then OC becomes very nearly equal to $OA-CA$, that is to $OA-BO$.

118. The truth of the parallelogram of forces when acting on a particle at rest may be roughly tested by some such apparatus as the following:

A board has three small smooth pulleys L, M, N fixed to it.



Three strings knotted together at O have their other extremities fastened to weights P_1, P_2, P_3; these strings are then arranged as in the figure, the plane of the board being vertical.

Now considering the equilibrium of the knot at O, we have three forces applied to it by the strings in the directions OL, OM and ON respectively, the forces being equal to the weights of P_1, P_2, P_3 respectively.

When the knot O has no acceleration the resultant of the three forces acting upon it must be zero. That is, any one of the forces must be the anti-resultant of the other two.

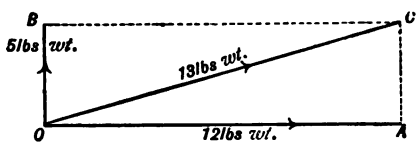
Suppose for example $P_1 = 3$ lbs. and $P_2 = 4$ lbs.; draw on the board lines OA, OB parallel to OM, ON making $OA = 3$ in., $OB = 4$ in., and complete the parallelogram $OAC'B$.

Then if we measure OC', it will be found that the number of inches in OC' will be equal to the number of pounds in P_3.

119. To find the resultant of **two forces at right angles to each other** we have to find the length and direction of the diagonal of a parallelogram, which in this particular case is a rectangle, whose sides are given.

Example. Find the resultant *of two forces of* 5 lbs. wt. *and* 12 lbs. wt. *at right angles to each other.*

Let OA represent 12 lbs. wt. and OB represent 5 lbs. wt., the angle AOB being a right angle.



Complete the parallelogram $OACB$ and join OC.

Then OC represents the required resultant.

Now $OC^2 = OA^2 + AC^2 = OA^2 + OB^2 = 12^2 + 5^2 = 169 = 13^2$.

Therefore OC represents 13 lbs. wt. ..... (i),

and the angle AOC is such that $\tan AOC = \frac{AC}{OA} = \frac{5}{12} = \cdot 416666 \ldots$ (ii).

Whence [from the Tables or by actual measurement] $AOC = 22° 28'$ nearly.

EXAMPLES. XXV.

Find the direction and magnitude of the resultant of the following pairs of forces; each pair being at right angles (working out the surds to 3 significant figures).

1. 3 lbs. wt. and 4 lbs. wt.
2. 15 pdls. and 10 pdls.
3. 25 lbs. wt. and 60 lbs. wt.
4. 6 pdls. and 7 pdls.
5. 10 lbs. wt. and 11 lbs. wt.
6. 10 pdls. and 20 pdls.
7. 4 lbs. wt. and 4 lbs. wt.
8. 1 pdl. and $\sqrt{3}$ pdls.

Find the acceleration produced by each pair of the above forces when applied in turn to the mass 100 lbs.

120. To find the resultant of **any two given forces** we may proceed as follows.

Let the forces P and Q inclined at the angle θ be represented by OA and OB.

Complete the parallelogram $OACB$. Let OC represent the force R. Draw CN perpendicular to OA then

$$AN = AC\cos\theta,\ NC = AC\sin\theta$$

$$OC^2 = ON^2 + NC^2 = (OA + AN)^2 + NC^2$$

$$= (OA + AC\cos\theta)^2 + AC^2\sin^2\theta$$

or

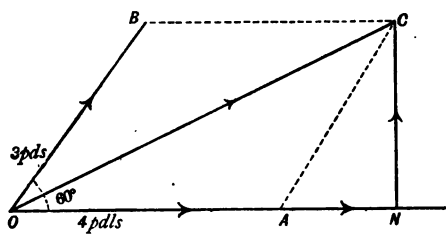
$$\mathbf{R^2} = (P + Q\cos\theta)^2 + Q^2\sin^2\theta$$

$$= P^2 + 2PQ\cos\theta + Q^2\cos^2\theta + Q^2\sin^2\theta$$

$$= \mathbf{P^2 + 2PQ\cos\theta + Q^2}.$$

Example. Find the resultant *of two forces of* 4 poundals *and* 3 pdls. *acting on a particle inclined to each other at an angle of* 60°.

Let OA represent 4 poundals, let OB represent 3 pdls., and let the angle $AOB = 60°$.



Complete the parallelogram $OACB$, then OC represents the required resultant.

We have to find the force represented by OC.

Let the unit length represent 1 poundal.

Then the length of $OB = 3$, the length of $OA = 4$.

Also

$$AN = AC\cos 60° = 3 \times \tfrac{1}{2},$$

$$NC = AC\sin 60° = 3 \times \tfrac{1}{2}\sqrt{3},$$

$$OC^2 = (OA + AN)^2 + NC^2 = (4 + 1\tfrac{1}{2})^2 + (\tfrac{3}{2}\sqrt{3})^2$$

$$= 1\tfrac{21}{4} + \tfrac{27}{4} = 37;$$

$$\therefore\ OC = \sqrt{37} = 6{\cdot}08\ldots \text{ pdls.}$$

EXAMPLES. XXVI.

N.B. *A* force *of* 5 lbs., *can only mean the* weight of the mass 5 lbs.

Find the resultants of the following forces.

1. Forces of 5 poundals and 5 pdls., inclined to each other at the angle whose cosine is $\frac{4}{5}$.

2. Forces of 6 lbs. and 3 lbs., inclined to each other at the angle whose cosine is $\frac{1}{2}$.

3. Forces of 7 lbs. and 12 lbs., inclined to each other at the angle whose cosine is $\frac{1}{7}$.

4. Forces of 3 lbs. and 4 lbs., inclined at the angle 30°.

5. Forces of 3 poundals and 8 pdls., inclined at 60°.

6. Forces of 5 lbs. and 10 lbs., inclined at 45°.

7. Forces of 1 lb. and $3\sqrt{2}$ lbs., inclined at 45°.

8. Forces of 10 lbs. and $10\sqrt{3}$ lbs., inclined at 150°.

9. A force of 20 lbs. towards the North, and a force of 30 lbs. towards the North-west.

10. A boat is pulled by a force of 30 lbs. towards the South and by a force of 20 lbs. towards the North-west. What other force is acting upon it if it remains at rest?

11. Two horses pull at a block of stone, one with a horizontal force of 100 lbs. the other with a horizontal force of 130 lbs., the forces being inclined to each other at the angle whose cosine is $\frac{12}{13}$; what is the force which can keep the stone at rest?

12. Two forces P lbs. and $\sqrt{2}P$ lbs. act at a point; P acts towards the East and $\sqrt{2}P$ towards the N.W.; find their resultant.

13. Two forces 2 lbs. and 4 lbs. inclined to each other at an angle of 120° act at a point. What is their resultant?

14. Two forces of 2 lbs. each acting at an angle of 60° have the same resultant as two equal forces acting at right angles; what is the magnitude of these two forces?

15. D is the middle point of the side AB of the equilateral triangle ABC; prove that the resultant of two forces represented respectively by AD, AC is represented in magnitude by $\sqrt{7}$ times AD.

16. Forces of 3 poundals and 4 poundals at right angles to each other act upon a mass of 5 lbs.; find the acceleration produced. How long will the mass take to move 18 ft. from rest?

17. Equal forces of 10 poundals at an angle of 120° to each other act on a mass of 1 lb.; how long will the mass take to acquire a velocity of 30 miles an hour?

18. Equal forces, each of the weight of 1 cwt., act on a ton placed on a smooth horizontal plane. The directions of the forces are respectively N. and W. If at the instant the forces begin to act the ton is at rest, where will it be and what velocity will it have after 5 seconds?

19. Two horses, attached by ropes to a mass of 1 ton placed on a smooth horizontal plane, pull at right angles to each other, one with a force equal to the weight of 3 cwt. and the other with a force equal to the weight of 4 cwt.; find the direction of motion and the acceleration of the mass.

20. If in Question **19** the plane is rough so that there is a force of friction which retards the motion equal to $\frac{1}{10}$th of the pressure of the mass on the plane; find the acceleration.

*CHAPTER IX.

RESOLUTES.

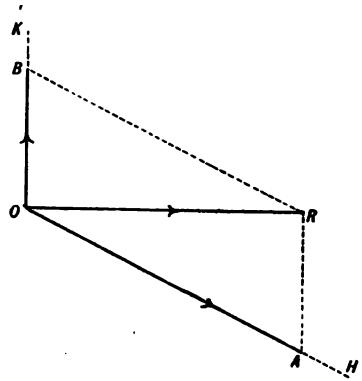
121. We may [Art. 114] at any time replace two forces which act upon a particle by a single force called their resultant; and the effect of this single force on the particle will be the same as the effect of the two forces together.

Conversely we may, if we please, at any time replace the single force (the resultant) by the two forces which together have the same effect.

122. When we replace a single force acting on a particle by two forces whose joint effect is the same, the two forces are called **components** of the single force.

123. PROP. *A force can be resolved into two components in any assigned directions.*

Let OR be the given force and OH, OK the given directions, from R draw RA parallel to KO and RB parallel to HO.



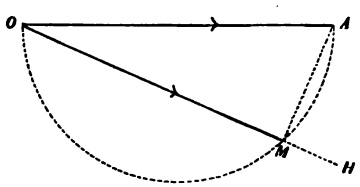
Then by the parallelogram of forces we may replace the force OR by the two forces represented by OA and OB.

124. DEF. When a force is resolved into two components at right angles to each other each component is called the **resolute** of the force in its own direction.

The **resolute of a force in a given direction** is an idea with which the student must make himself perfectly familiar.

125. *To find the* **resolute** *of a force in a given direction.*

Let OA represent the given force, OH the given direction;



from A draw AM perpendicular to OH; then OM represents the required resolute in the direction OH.

126. When the given direction makes an angle α with the given force so that $AOH = \alpha$,

then, since $$\frac{OM}{OA} = \cos\alpha,$$

we have $$OM = OA\cos\alpha,$$

or, the resolute of a force P in a direction making the angle α with it is $$P\cos\alpha.$$

Also the resolute of OA perpendicular to OM is $$OA\sin\alpha.$$

NOTE. That which we have here called the *resolute* of a force is in many books on Mechanics called **the resolved part** of a force.

The name *Resolute* (first suggested in my *Elementary Statics*) is mentioned with approval by Dr Routh in his recent treatise on *Statics*. Cambridge University Press, 1891, p. 26.

Example i. *Find the* resolute *of a force of* 10 poundals *in the direction making the* angle 60° *with it.*

Here OA represents 10 pdls. and $AOM = 60°$,
therefore $OM = 10\cos 60° = 5$ pdls.

Example ii. *Find the* resolute *of a force of* 321 lbs. wt. *in the direction making the* angle whose cosine is ·647 *with it.*

Here OA represents 321 lbs. wt. and $\cos AOM = ·647$,
therefore $OM = 321 \times ·647$.

EXAMPLES. XXVII.

Find the resolute of each of the following forces in the direction making the given angle with it.

1. 30 poundals., at the angle 60° with it.
2. 50 lbs. wt., at the angle 30° with it.
3. $4\sqrt{2}$ lbs. wt., at the angle 45° with it.
4. 125 lbs. wt., at the angle whose cosine is $\frac{4}{5}$ with it.
5. 300 lbs. wt., at the angle whose cosine is ·7 with it.
6. 437 lbs. wt., at the angle whose cosine is ·125 with it.
7. 237 lbs. wt., at the angle whose cosine is ·794 with it.
8. 34·7 lbs. wt., at the angle 35° 48′ 30″ [cos 35° 48′ 30″ = ·811].

Find the following resolutes by means of a scale and protractor.

9. 25 lbs. wt. at the angle 36°.
10. 3 lbs. wt. at the angle 45° 30′.
11. 22 lbs. wt. at the angle 40°.
12. 12 lbs. wt. at the angle 52°.
13. 18 lbs. wt. at the angle 20°.
14. 10 lbs. wt. at the angle 25°.

127. As the student proceeds in the subject he will find that *the resolute* of a given force *in a given direction* may be said to be that which represents **the effect** of the given force **in that direction.**

The same remark is true of the resolute of every kind of directional quantity; it is true of *distance*, of *displacement*, of *velocity*, of *acceleration*, of *impulse.*

Example. A force may be said to have no effect in the direction perpendicular to itself. The resolute of a force in the direction perpendicular to the force is zero. A force may have a *component* of any given magnitude in the direction perpendicular to itself.

128. By Art. 126 the resolute of a force P lbs. in the direction making the angle α with it, is $P\cos\alpha$ lbs.

We notice that $\cos\alpha$ is a *number;* so that $P\cos\alpha$ lbs. is a *force.*

We notice that $\cos\alpha$ is never greater than 1, so that the resolute of a force is always less than the force itself,

Except when it is in the direction of the force, and then it is the force itself.

In the figure of Art. 125, if a circle be described on OA as diameter, the chord of this circle drawn through O in any direction represents the resolute in that direction.

129. The student must notice that two separate resolutes of the same force are not a pair of *components* of the force unless these two resolutes are *at right angles to each other.*

130. There is one direction and one direction only in which the resolute of a force is zero; namely, the direction perpendicular to the force itself.

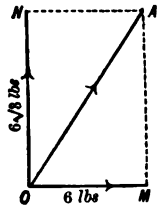
Hence no force can have its resolutes in two different directions each separately zero. A fact which may be stated thus;

When the resolutes of a force in two different directions are each separately zero that force must be itself zero.

This fact is most important, and will often be used in subsequent parts of the book.

Example i. *A force has its* resolute *towards the North equal to* $6\sqrt{3}$ lbs., *the* other component *is* 6 lbs. *towards the East; find the magnitude and direction of the force.*

Let OM represent the force of 6 lbs. and ON that of $6\sqrt{3}$ lbs.



Complete the rectangle $NOMA$.

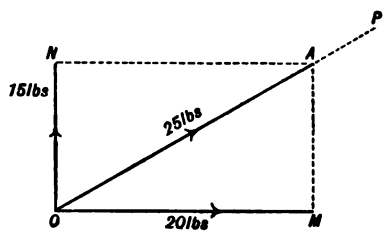
Then OA represents the required force and

$$OA = \sqrt{\{6^2 = 6^2 \times 3\}} \text{ lbs.} = \underline{12 \text{ lbs.}}$$

Also, since $\dfrac{OM}{OA} = \dfrac{6}{12} = \dfrac{1}{2}$, the angle $\underline{MOA = 60°}$.

Example ii. *The resolute of a force in the direction making the* angle whose sine is $\frac{4}{5}$ *with the force is* 20 lbs.; *find the* force *and the* other component *of the force.*

Let OM represent the given resolute.



Draw the angle MOP whose sine is $\frac{4}{5}$, draw MA perpendicular to OM cutting OP in A.

Then OA represents the required force.

Complete the rectangle $OMAN$; then ON is the other component of the force.

Now, since $\sin MOA = \dfrac{3}{5}$, $\therefore \dfrac{AM}{OA} = \dfrac{3}{5}$ and $\dfrac{OM}{OA} = \dfrac{4}{5}$;

$$\therefore OA = \tfrac{5}{4} \text{ of } OM = \tfrac{5}{4} \text{ of } 20 \text{ lbs.} = \underline{25 \text{ lbs.}},$$

also $$MA = \tfrac{3}{4} \text{ of } OM = \tfrac{3}{4} \text{ of } 20 \text{ lbs.} = \underline{15 \text{ lbs.}}$$

EXAMPLES. XXVIII.

1. Find the resultant of forces of 4 lbs. and 3 lbs. inclined to each other at an angle of 45°.

2. The rectangular components of a force in magnitude are 3 lbs. and 4 lbs. respectively; what is the force?

3. The rectangular components of a force are each 5 lbs.; what is the force?

4. The resolute of a force which makes an angle of 60° with the force is 10 lbs.; what is the force?

5. The resolute of a force which makes an angle of 30° with it is $\sqrt{3}$ lbs.; what is the force? and find the other component.

6. The resolute of a force making an angle of 45° with it is 20 lbs.; what is the other component?

7. The resolute of a force making an angle whose tangent is $\frac{3}{4}$ with the force is 12 lbs.; find the other component.

8. If two forces are represented in magnitude and direction by the side (of length l ft.) and a diagonal of a square, prove that their resultant will be represented either by a line of length $\sqrt{5}l$ or by a line of length l.

9. ABC is an equilateral triangle; forces P lbs., P lbs., Q lbs. act at a point parallel to AC, CB, AB respectively; shew that their resultant is $(P+Q)$ lbs.

10. Forces of 25 lbs., 24 lbs. and 7 lbs. acting at a point are in equilibrium; shew that two of them are at right angles.

11. Forces of 40 lbs., 41 lbs. and 9 lbs. acting at a point are in equilibrium; shew that two of them are at right angles.

12. A force of 10 lbs. acts along the side AB of an equilateral triangle; what is the resolute of this force along the side AC? find also its resolute in the direction parallel to CB.

13. Shew that the resultant of the two resolutes in **12** is 5 lbs.

14. Prove that if OA represent a force and a circle be described on AB as diameter, the resolute of the force in any direction is represented by the chord of the circle, drawn in that direction through O.

CHAPTER X.

MOMENTS.

131. DEF. The **moment** *of a force about a point* is that which varies as the force and also as the perpendicular distance of the point from the line of action of the force.

That is, if the force is doubled or trebled...the moment of the force about a given point is doubled or trebled...; also if the perpendicular distance of the point from the line of action of a given force is doubled or trebled...the moment of the given force about the point is doubled or trebled.

$\therefore$ Its moment $= K \times$ the force $\times$ perpendicular.

132. We shall assume as our **unit** *moment* that of unit force about a point whose distance from the line of action is the unit distance.

The unit force being 1 lb. weight, the unit distance being a foot, the unit moment would be a **foot-pound weight.**

Hence the moment of a force about a point is *measured* by the product of the number of lbs. in the force by the number of feet in the perpendicular distance of the line of action of the force from the point; or shortly,

The moment of a force about a point is *numerically equal* to the product of the force into its perpendicular distance from the point.

Example. Three forces *of* 3 lbs., 4 lbs. *and* 5 lbs. *act along the two sides* AB, AD *and the diagonal* AC *of the square* ABCD *each of whose sides is* 2 ft.; *find the* measures of the moments *of each of these forces about the point* D.

The moment of the force 3 lbs. along AB is numerically equal to

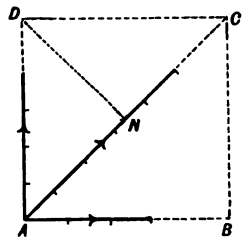
$$3 \times AD = 3 \times 2 = 6.$$

The moment of the force 5 lbs. along AC is numerically equal to $5 \times DN$, where DN is perpendicular to AC.

$$DN^2 + AN^2 = AD^2 = 4 \text{ and } DN = AN;$$

$$\therefore\ DN^2 = 2\ ;\ \therefore\ DN = \sqrt{2},$$

the required measure $= 5 \times \sqrt{2} = 7{\cdot}071\ldots$



The moment of the force along AD about D is zero; for the perpendicular from D on AD is zero.

EXAMPLES. XXIX.

1. $ABCD$ is a square each of whose sides is 5 feet; forces of 5 lbs., 6 lbs., and 7 lbs. act along AB, AC, and AD, find the moment of each of these forces about D.

2. Find the moment of each of the forces of Question **1** about the middle point of AC.

3. Find the moment of each of the forces of Question **1** about the middle point of DC.

4. $ABCD$ is a square each of whose sides is 18 inches; forces o 3 lbs., 4 lbs., 5 lbs. and 6 lbs. act along AB, BC, CD and DA respectively; find the moment of each of these forces about a point Q in AD produced so that $DQ = 6$ in.

5. Find the moment of each of the forces in Question **4** about a point Q on AC produced so that $CQ = 6\sqrt{2}$ inches.

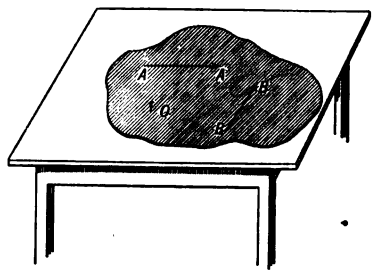
6. A and B are two points 1 foot apart; a force of 5 lbs. acts at A perpendicular to AB and a force of 7 lbs. acts at B parallel to the first force; find the point in AB about which the moments of these forces are equal in magnitude.

7. Find the point in AB produced about which the moments of the two forces in Question **6** are equal.

8. ABC is an equilateral triangle and forces of 4 lbs. and 5 lbs. act at A along AB and AC respectively, find the point in BC about which the moments of these forces are equal.

133. The following experiment is important.

Take a thin piece of board having a small hole in it at O. Let the board be placed on a smooth horizontal table, and let a small tack be



driven into the table at the point O so that the board can turn freely about O.

Let two small tacks be fastened to the board at any two points A and B, and let two horizontal forces represented by AA', BB' be applied (by strings fastened to the tacks) to the board at A and A'.

It will be found that when the board is in equilibrium under the action of these forces,

(i) the moment of the force AA' about O is equal to the moment of the force BB' about O.

(ii) that if each of the forces in them be allowed to act alone for a moment they will each exhibit a power of causing the board to turn round O, but *in opposite senses.*

These two conditions are expressed shortly thus. *When a board one point* O *of which is fixed, is in equilibrium the moments about* O *of the forces which act upon the board must be equal in magnitude and of opposite senses.*

134. The manner in which the force AA', as drawn above, tends to produce rotation about O is said to be **clockwise**; the manner in which the force BB', as drawn above, tends to produce rotation is said to be **counter-clockwise.**

Being respectively like and unlike the rotation of the hands of a clock when laid face upwards on a table.

135. It will be found convenient to indicate that two moments are of the same or of contrary senses by the aid of the signs + and −.

And in what follows when use is made of the **sum** of the moments about a point of any number of forces, the sum will be understood to mean their algebraical sum, the moment of each force having its proper sign according to its sense.

The necessary condition that the board in the experiment of Art. 133 should be in equilibrium may now be stated thus. The sum of the moments of the two forces acting upon it about the point *O* must be zero.

It must be clearly understood that the experiment of Art. 133 does not *prove* this statement. Its truth will be deduced later on from the laws of motion.

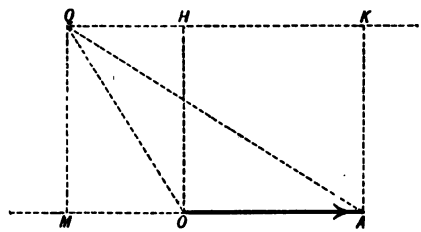
It is perhaps the more usual to consider counter-clockwise rotation as positive.

136. Since the moment of a force about a given point is numerically equal to the *product* of the force into the perpendicular distance from the point to the line of action of the force, it follows that

The moment of a force about a point is *numerically equal* to twice the area of the triangle formed by joining the point to the extremities of the line which represents the force.

[Provided the unit force is represented by the same unit of scale as the unit distance.]

For let OA represent a given force in magnitude, direction and line of action; let Q be any chosen point.



Draw QM perpendicular to the line of OA.

Then the moment of the force OA about Q is measured by the **product** of the *number* of lbs. in the force OA, by the *number* of linear units in QM.

But the number of lbs. in the force is the number of linear units in OA.

Hence the moment of the force OA about the point Q is numerically equal to $QM \times OA$.

Through Q draw QHK parallel to OA, and draw OH, AK perpendicular to OA.

Then $QM \times OA = HO \times OA$.

And $HO \times OA$ numerically equals the area of the rectangle $HOAK$, which is double of the area of the triangle QOA,

$\therefore$ $QM \times OA =$ double of the area of the triangle QOA.

EXAMPLES. XXX.

1. Prove that the sum of the moments of two equal parallel and opposite forces, not in the same line of action about any point in their plane, is the same for all positions of that point.

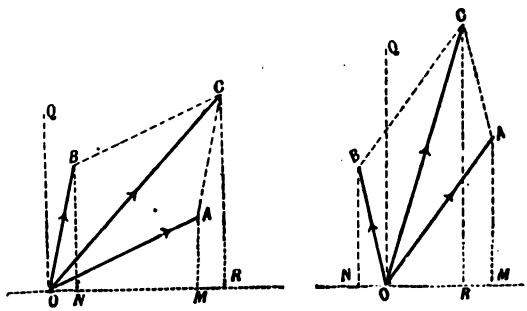
2. Forces are represented in position, magnitude and sense by the sides of a closed polygon taken the same way round; prove that the sum of the moments of these forces about any point in their plane is numerically equal to twice the area of the polygon.

3. Three forces act along the sides of an equilateral triangle taken the same way round; shew that the sum of their moments about a point in their plane is different for different positions of the point, except when the forces are equal.

4. Prove that the moment of a given force about each of three points cannot be zero unless the three points lie in the same straight line.

137. PROP. ***The sum of the moments of two forces acting at a point about any given point is equal to the moment of their resultant about the same point.***

Let OA, OB be the forces; OC their resultant.



Let Q be the given point.

Then we have to prove that the area of
twice ($\triangle QOA + \triangle QOB$) = twice $\triangle QOC$.

Draw a line ONM at right angles to OQ.

Draw AM, BN, CR each parallel to QO.

Then, twice $\triangle QOA$ = the rectangle $OM \times OQ$,
twice $\triangle QOB$ = the rectangle $ON \times OQ$,
twice $\triangle QOC$ = the rectangle $OR \times OQ$.

Now OM, ON, OR are the resolutes of the forces OA, OB, OC in the direction OM, and since AC is equal and parallel to OB, MR is equal to ON,

$$\therefore\ OM + ON = OM + MR = OR,$$
$$\therefore\ OM \times OQ + ON \times OQ = OR \times OQ.$$

That is

twice $\triangle QOA$ + twice $\triangle QOB$ = twice $\triangle QOC$. Q. E. D.

138. PROP. *The sum of the moments of a number of forces acting at a point in a plane about a given point in their plane is equal to the moment of their resultant about the same point.*

For the sum of the moments of any two of the forces about the point is equal to the moment of their resultant about that point.

The sum of the moments of this resultant and a third force about the point is equal to the moments of *their* resultant about the same point;

and so on.

So that the sum of the moments of all the forces about the point is equal to the moment of the last resultant about the same point; the last resultant being the resultant of all the forces. Q. E. D.

139. PROP. *When a number of forces in one plane acting at a point are in equilibrium, then the sum of their moments about every single point in their plane is zero.*

For the sum of the moments of the forces about the point is equal to the moment of their resultant about the same point.

The resultant of the forces is *zero;* for the forces are in equilibrium.

Therefore the moment of the resultant about any point is zero.

140. There is always a series of points about any one of which the sum of the moments of any given system of forces acting at a point is zero; namely, the series of points which lie in the line of action of the resultant of the given system of forces.

For the sum of the moments of a given system of forces about any chosen point Q is equal to the moment about Q of their resultant.

When Q is on the line of action of this resultant, the perpendicular QM drawn from Q to this line of action is zero; and therefore the product $QM \times OA$ is, in this particular case, zero.

141. PROP. *The moments of a force about each of three points not on the same straight line cannot be each separately zero.*

For the moment of a force about a point is only zero when its line of action passes through the point; and a straight line cannot pass through each of three points not in the same straight line.

142. PROP. *When a system of forces acting at one point in a plane are such that the sum of their moments about each of three points not in the same straight line, is each separately zero, the system of forces must be in equilibrium.*

For the sum of the moments of the forces about any one point Q is equal to the moment about Q of their resultant.

Therefore the moments of the resultant about each of three points not in the same straight line are each separately zero.

This by Art. 131 is impossible unless the resultant is itself zero. That is, the system of forces is in equilibrium.

Q. E. D.

Example. It is observed of a system of forces acting at a point that the sums of their moments about each of two given points are equal. What does this prove of their resultant?

The resultant of the forces either (i) is parallel to the line joining those two points or (ii) passes through the point of bisection of the line joining the two given points. For its perpendicular distances from each of these two points must be equal.

EXAMPLES. XXXI.

1. It is observed that the sums of the moments of two forces acting at a point about each of two points in their plane are in the ratio of 2 to 1; what do you infer with regard to the resultant of the two forces?

2. The sum of the moments of certain forces acting at a point O is found to be zero about a certain point Q; what can be inferred from that fact with regard to their resultant?

3. Shew that the sum of the moments of the forces in Question **2** about any point in the line OQ is zero.

4. If the sum of the moments of the forces in Question **2** is also zero about some point not in the line OQ, shew that the forces must be in equilibrium.

5. Forces of 3 lbs. and 4 lbs. act along the sides AB, AD of a square $ABCD$ such that $AB=5$ ft.; shew that the distance of their resultant from the point B is 4 ft., and from the point C is 1 ft. and from the point D is 3 ft.

COUPLES.

143. DEF. A **couple** consists of two equal parallel forces of opposite senses, not in the same line of action.

When a workman applies to the head of a screw a twist (without pressing in the direction of the screw) he applies *a couple.*

When a man winds up his watch or a clock with a symmetrical key he does it by means of *a couple.*

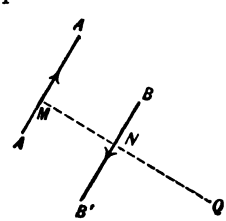
NOTE. Parallel forces are said to be **like** or **unlike** according as they act in the *same* or in *opposite* senses.

144. PROP. *The sum of the moments of the two forces forming a couple, about any point in their plane has a* constant value; *this value is numerically equal to* the product of one of the forces into the perpendicular distance between them.

Take any point Q in the plane containing the two equal parallel and opposite forces AA', BB'.

Draw QMN perpendicular to the lines of action of the forces cutting them in M and N.

Then, the sum of the moments of these two forces about Q is numerically equal to



$$\begin{aligned} & QN \times BB' - QM \times AA' \\ &= (QM - QN) \times AA' \qquad [\text{for } AA' = BB'] \\ &= NM \times AA', \end{aligned}$$

which is independent of the position of Q.

145. DEF. The **moment of a couple** is numerically equal to the product of one of the forces into the perpendicular distance between the forces.

146. The moment of two equal parallel unlike forces is not zero except:

(i) when the forces are each zero;

(ii) when they are in the same line of action;

that is to say, when the two forces do not form a couple.

Hence, **the moment of a couple is never zero.**

Example. Three forces, not parallel to each other, are such that if they acted at one point they would be in equilibrium; prove that they are equivalent to a couple.

The resultant of two of the forces must be equal, parallel and of opposite sense to the third force; for if this resultant and the third force acted at a point they would be in equilibrium.

This resultant and the third force therefore form a couple.

EXAMPLES. XXXII.

1. Four forces in the same plane not all parallel are such that if they acted at one point they would be in equilibrium; prove that they are equivalent to a couple.

2. Three forces are represented in direction, magnitude, and line of action by the sides of a triangle taken the same way round. Prove that they form a couple.

3. Prove that the moment of the couple of Question **2** is numerically equal to twice the area of the triangle.

4. Forces are represented in direction, magnitude, and line of action by the sides of a square taken in order; prove that the sum of their moments about each point in the plane of the square is constant.

5. Prove that the forces in Question **4** are equivalent to a couple.

6. Prove that the moment of the couple in Question **5** is numerically equal to twice the area of the square.

SECTION IV.

PARALLEL FORCES.

CHAPTER XI.

FORCE ACTING ON RIGID BODIES.

147. We have hitherto supposed that the forces acting on a body act at the same point of a body.

In the explanation of the moment of a force we suggested *experiments* in which forces did not so act, but we carefully pointed out that we gave no proof of the result of the experiments.

The discussion of the *motion* of bodies which cannot be treated as *particles*, and whose rotation must therefore be considered, is beyond the scope of this work.

The particular case however of a rigid body, which is at rest and although acted on by **parallel** forces in one plane continues at rest, is capable of the following simple treatment.

148. DEF. A **body** is a limited portion of matter which we consider as composed of very small particles.

149. DEF. A **rigid** body is a body whose changes of shape, due to the action on it of any available forces, are too small to be of practical importance.

It must be noticed that force applied to a body however rigid, sets up **strains** within the body; and that these strains are slight changes of shape or of size.

150. A **rod** is a *rigid* body whose length is so great compared with its breadth and its thickness that the last two (breadth and thickness) need not be considered.

A **light** *rod* is a rod whose *weight* is inconsiderable compared with the other forces which act upon it.

A **lamina** is a *rigid* body whose surface is so great compared with its thickness that the latter (thickness) need not be considered.

A **string** is a body whose breadth and thickness are small compared with its length; which is incapable of exerting any stress except in the direction of its length; this stress is of one kind only, namely, a '*pull.*'

Thus a string is *rigid* in one direction only.

151. We shall make use of the following principle.

When a force acts upon a body which is rigid, **the point of application** of the force (for the purpose of calculating the conditions of equilibrium of the body) may be considered to be **any point in its line of action.**

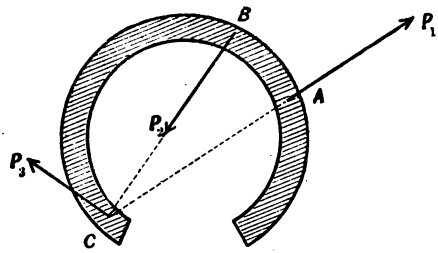
This is called the Principle of the **Transmissibility of Force.**

152. Consider a body at rest acted on by three forces whose points of application are not all the same. Let the lines of action of the forces all pass through a point outside the body.

Suppose this point O rigidly connected with the body by a system of fine rigid wires.

Then by the principle of the Transmissibility of Force we may suppose all the forces to be applied to the rigid body at O, and therefore if one of them is the anti-resultant of the other two as found by the parallelogram law, the forces will be in equilibrium and the body will have no tendency to move.

For example, consider a body ABC in the shape of a horse shoe. Let forces P_1, P_2, P_3 act upon it as in the figure, at the points A, B, C respectively.



Let the line of action of each force pass through *C*.

It will be found that when the forces are in equilibrium each is the anti-resultant of the other two.

Moreover it will be found that if the line of P_1 passes through *C* that there cannot be equilibrium unless the line of P_2 also passes through *C*.

153. PROP. *To find the condition for the equilibrium of a system of* **parallel forces** *acting in the same plane upon a rigid body.*

When a rigid body is acted on by a system of forces in one plane whose lines of action meet at a point the necessary and sufficient condition of equilibrium is that their resultant, as found by the parallelogram law, should be zero. And the resultant is necessarily zero if the moment-sums of all the forces about each of three points not in the same straight line, are each zero.

And this is true even if the point in which the lines of action of the forces meet is not within the body. It is therefore true when the forces are such that the point in which their lines meet is at a great distance from the body.

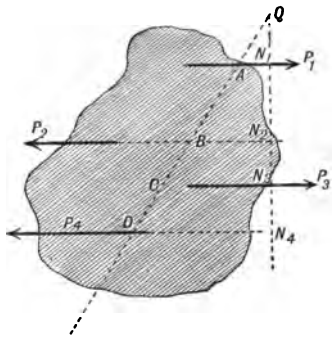
This is true no matter how great be the distance of the point of meeting of the lines of action of the forces.

And therefore it is also true when the point at which they meet is at an infinitely great distance so that the lines of action are actually parallel.

Hence the required condition that the system of parallel forces should be in equilibrium is that **the moment-sums of the forces about each of three points in their plane should be zero, the three points not being in a straight line.**

N.B. In calculating a moment-sum, due regard must be paid to the *sense* of each moment.

154. Let P_1, P_2, P_3, P_4... be any system of parallel forces acting in one plane on a rigid body.



Draw the line $N_1N_2N_3N_4$ perpendicular to the forces cutting the lines of action of the forces P_1, P_2, P_3, P_4... in the points N_1, N_2, N_3, N_4...

Then if the system of forces P_1, P_2, P_3, P_4,... is in equilibrium we have that their moment-sum about any point Q in N_1N_2 is zero,

$$\therefore P_1 \times QN_1 + P_2 \times QN_2 + P_3 \times QN_3 + P_4 \times QN_4 + \ldots = 0.$$

Q may if we please coincide with N_1,

$$\therefore P_1 \times 0 + P_2 \times N_1N_2 + P_3 \times N_1N_3 + P_4 \times N_1N_4 + \ldots = 0.$$

Subtracting, we have

$$P_1 \times QN_1 + P_2 \times (QN_2 - N_1N_2) + P_3 \times (QN_3 - N_1N_3) + P_4 \times (QN_4 - N_1N_4) + \ldots = 0$$

$$\text{or } QN_1 \times (P_1 + P_2 + P_3 + P_4 + \ldots) = 0.$$

Therefore when a system of parallel forces is in equilibrium the sum of the forces P_1, P_2, P_3, P_4,... is zero.

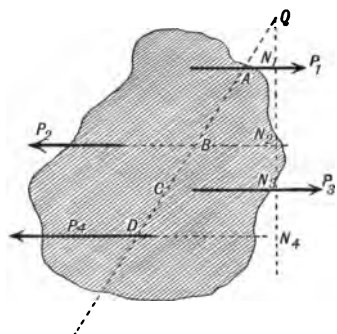
155. The result of Art. 154 will be more evident when the student considers that a system of parallel forces is the limiting case of parallel forces acting at one point, when this point is at a very great distance.

156. The necessary and sufficient condition for the equilibrium of a system of parallel forces in one plane acting upon a rigid body may be stated thus:

(i) **The sum of the forces must be zero.**

(ii) **The moment-sum of the forces about a point in their plane must be zero.**

For with the notation of Art. 154 let Q be any point in the line $N_1N_2N_3$.



The moment-sum of the forces about Q is

$$P_1 \times QN_1 + P_2 \times QN_2 + P_3 \times QN_3 + P_4 \times QN_4 + \ldots;$$

let this be zero. Also let

$$P_1 + P_2 + P_3 + P_4 + \ldots \text{be zero.}$$

Let Q' be any other point in the line $N_1N_2N_3$.

Then $P_1 \times (Q'Q + QN_1) + P_2 \times (Q'Q + QN_2) + P_3 \times (Q'Q + QN_3)$
$+ P_4 \times (Q'Q + QN_4) + \ldots = 0.$

That is the moment-sum of P_1, P_2, P_3, P_4... about any other point in their plane is zero.

Hence when the above two conditions are satisfied, the conditions of Art. 153 are satisfied. Q. E. D.

157. From Q draw a line in any direction cutting the lines of action of the forces in A, B, C....

Then $$\frac{QN_1}{QA} = \frac{QN_2}{QB} = \frac{QN_3}{QC} = \ldots.$$

Let the common value of these fractions be k, then

$QN_1 = k \cdot QA$, $QN_2 = k \cdot QB$, $QN_3 = k \cdot QC$, etc.,

whence equating the moment-sum of the forces about Q to zero (as in Art. 156), and dividing by k,

$$P_1 \times QA + P_2 \times QB + P_3 \times QC + \ldots = 0 \ldots\ldots \text{(ii)}.$$

Hence in using Art. 156, II, in the case of *parallel* forces, we may write down 'the equation of moments' thus,

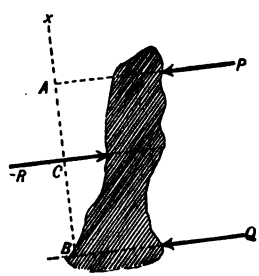
$$P_1 \times QA + P_2 \times QB + P_3 \times QC + \ldots = 0.$$

The statement is true even when the line $QABC$ is not perpendicular to the line of action of the forces.

That is, when we *take the moments of a system of* **parallel** *forces* about a point, the distances of the forces from the point may be measured in any one direction which may be convenient.

Example. When there are three parallel forces in equilibrium, viz. P, Q, *and* − R, *then if a line cut these forces in* A, B, C *respectively,*

$$P \times AC = Q \times BC.$$



Taking moments about any point X, we have

$$P \times AX + Q \times BX - R \times CX = 0,$$

also $$R = P + Q,$$

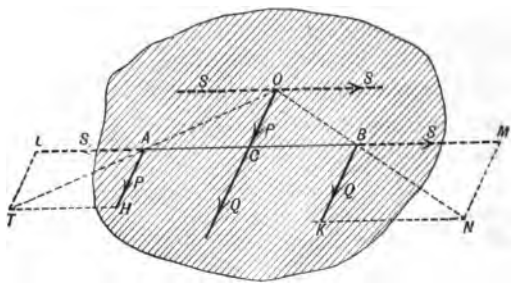
$$\therefore\ P \times AX + Q \times (BC + CA + AX) - (P + Q)(CA + AX) = 0.$$

$$\therefore\ Q \times BC = P \times AC.$$

*158. We proceed to give an independent investigation of the result of Art. 156 for two parallel forces.

To find the resultant of two like parallel forces acting on a rigid body.

Let A, B be the points of application of the two parallel forces P and Q; let AH, BK represent them in direction and magnitude.



Let two equal and opposite forces each of magnitude S be applied to the rigid body along the line AB.

Let AL represent one force S and BM represent the other force S.

Since the body is rigid these two forces will not alter the external effect of the forces P and Q. [Art. 185.]

Complete the parallelograms $HALT$, $KBMN$.

Then, since the angles HAB, KBA are together equal to two right angles, the angles TAB, NBA are together greater than two right angles. Therefore TA and NB will if produced meet on the other side of AB.

Let them meet at O.

Draw OC parallel to AH or BK, to cut AB in C.

Now the two forces AH, AL may be replaced by the force AT; which may be supposed to act at O; and may there be replaced by two forces equal and parallel to AH and AL; that is, to P and S.

Similarly, the two forces BK, BM may be replaced by the force BN; which may be supposed to act at O; and may there be replaced by two forces equal and parallel to BK, BM; that is, to Q and S.

Thus finally, instead of the two forces P, Q applied to the rigid body at A and B respectively, we now have the four forces applied at O; namely two equal and opposite forces each equal to S, and two like forces P and Q acting in the line OC; the two forces each equal to S are in equilibrium and may be omitted.

Hence, the resultant of the two like parallel forces, is a single force acting in the line OC parallel to the two forces, and applied at any point in OC; also its magnitude is $P+Q$.

Now the triangles HAT and COA are similar,

$$\therefore \quad \frac{AH}{HT}=\frac{OC}{CA}; \text{ that is, } \frac{P}{S}=\frac{OC}{CA}.$$

Also the triangles KBN, COB are similar,

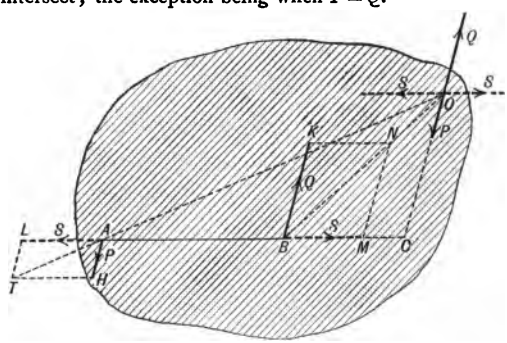
$$\therefore \quad \frac{KN}{BK}=\frac{CB}{OC}; \text{ that is } \frac{S}{Q}=\frac{CB}{OC},$$

$$\therefore \quad \frac{P}{S}\times\frac{S}{Q}=\frac{OC}{CA}\times\frac{CB}{OC}.$$

Hence, $$\frac{P}{Q}=\frac{CB}{CA}; \text{ that is, } P\times CA=Q\times CB. \qquad \text{Q. E. D.}$$

159. PROP. *To find the resultant of two unlike parallel forces acting on a rigid body.*

With a similar construction, the proof is word for word the same as that of the last proposition; except that AT and BN do not in this case *always* intersect; the exception being when $P=Q$.



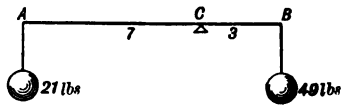
The resultant will be the '*sum*' of the two forces; which, as the forces are unlike, will *numerically* be equal to their difference, and will act in the direction of the greater force.

The figure given is for the case in which Q is greater than P.

For when Q is greater than P, since $BM=AL$, the angle NBM is greater than LAT; that is, than OAB; so that OBA is correctly drawn as a *triangle*.

When the forces P and Q are equal and unlike the lines AT, NB in the above figure are always parallel, and the construction fails. In fact, in this case P and Q together form a couple and are not capable of a single resultant.

Example i. *A light rod* 10 ft. *long has weights of* 21 lbs. *and* 49 lbs. *fastened one at each end. A man wishes to carry them both; where must he take hold of the rod and what weight will he lift?*



Here, considering the equilibrium of the rod, we have two like parallel forces acting upon it. Their resultant is therefore equal to their sum.

The man must therefore lift (21 + 49) lbs. = 70 lbs.

Also if AB be the rod and C be the point at which the resultant acts (which is the point at which the man must take hold of the rod), we have

$$AC + CB = 10 \text{ ft.}$$

and

$$21 \times AC = 49 \times CB,$$

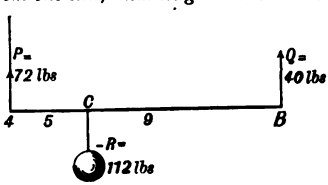
$$\therefore\ 49\,AC + 49\,BC = 490 \text{ ft.}$$

$$\therefore\ 49\,AC + 21\,AC = 490 \text{ ft.}$$

$$\therefore\ AC = 7 \text{ ft.}$$

and $BC = 3$ ft.

Example ii. *Two men carry a weight of* 1 cwt. *slung on a light pole* 14 ft. *long, each holding one end of the pole; if the weight be placed at a point* 5 ft. *from one end, what weight does each man carry?*



Consider the forces acting on the rod.

In this case $-R$ is 112 lbs. downwards; P and Q are the forces applied by the men's hands to the rod; the weight of the rod itself is neglected.

AC is 5 ft., BC is 9 ft.;

$$P + Q = R = 112 \text{ lbs. upwards} \quad \text{.....(i).}$$

Also

$$P \times 5 = Q \times 9 \quad \text{.....(ii),}$$

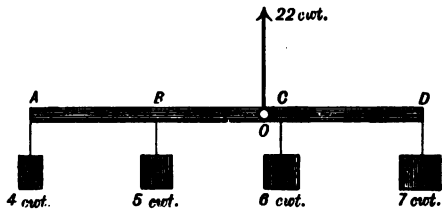
$$\therefore\ 5P + 5Q = 112 \text{ lbs.} \times 5 = 560 \text{ lbs.}$$

$$\therefore\ 9Q + 5Q = 560 \text{ lbs.}$$

$$\therefore\ Q = \underline{40} \text{ lbs.}$$

and $P = (112 \text{ lbs.} - Q) = \underline{72}$ lbs.

Example iii. *Weights of* 4 cwt., 5 cwt., 6 cwt., 7 cwt. *are suspended from points* ABCD *of a light horizontal rigid rod which are* 1 foot *apart; find* where *a single upward force must be applied to the rod that it may be in equilibrium.*



The rod is *light;* this means that we are to neglect its weight. The only forces acting on the rigid body, the rod, are the weights; these weights are vertical forces acting downwards.

Since the sum of the resolved parts in any direction of a system of forces in equilibrium is zero, we see, by considering the resolved parts in the vertical direction, that the upward force must

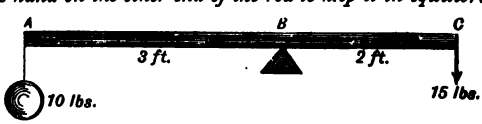
$$= (4+5+6+7) \text{ cwt.} = 22 \text{ cwt.}$$

By taking the sum of the moments about the point A of all the forces, we have, if OA contain x feet,

$$4 \times 0 + 5 \times 1 + 6 \times 2 + 7 \times 3 - 22 \times x = 0,$$

or
$$x = \tfrac{38}{22} = 1\tfrac{8}{11} \text{ ft.} = OA.$$

Example iv. *A light rigid rod* 5 ft. long *has a* weight of 10 lbs. *suspended at one end and is supported at a point* 3 ft. *from that end by the upward pressure of a man's shoulder; what* force *must be exerted by the man's hand on the other end of the rod to keep it in equilibrium?*



Let AC be the rod, B the position of the man's shoulder.

Let x lbs. be the vertical pressure upwards on the rod at B.

Consider the equilibrium of the rod.

Then since the forces of 10 lbs. and x lbs. are parallel, and the force of x lbs. is the resultant of two like parallel vertical forces acting downwards, x lbs. = the sum of these forces. Therefore the other force is $(10x - x)$ lbs.; hence

By taking the sum of the moments about C of all the forces we have

$$10 \times AC - x \times BC + (x - 10) \times 0 = 0,$$

or
$$10 \times 5 - x \times 2 = 0;$$

$$\therefore \quad x = 25.$$

Hence the required force, which is $(x - 10)$ lbs., is $(25 - 10)$ lbs., or 15 lbs. downwards.

EXAMPLES. XXXIII.

1. Find the resultant of two like parallel forces of 3 lbs. and 2 lbs. acting at points 5 ft. apart.

2. Find where a force must be applied to a rigid bar to keep it in equilibrium when acted on by two like parallel forces of 3 lbs. and 5 lbs. acting at points on the bar 4 ft. apart.

3. Two men carry a weight of 1 cwt. by means of a light rigid rod 14 ft. long, each having one end of the rod on his shoulder: find what each man carries when the weight is suspended from a point 6 ft. from one end.

4. Two men carry a barrel weighing 80 lbs. by means of a rigid pole 10 ft. long, each supporting one end of the rod and the barrel is slung at a point on the rod 4 ft. from one end: what weight does each man carry?

5. A man carries two weights by means of a rod 12 ft. long supported by his shoulder; if the rod presses on his shoulder with a force of 36 lbs., and the point of the rod on his shoulder is 4 ft. from one end; what are the two weights?

6. A man carries two buckets of water by means of a pole which he holds in his hand at a point three-fifths of its length from one end; if the total weight carried is 40 lbs., how much do the buckets weigh respectively?

7. A rigid bar 4 ft. long is acted on by three like parallel forces of 3 lbs., and 3 lbs. at each end and one of 4 lbs. at the middle point; what force is necessary to maintain equilibrium?

8. A rigid bar 4 ft. long is acted on by equal like parallel forces of 4 lbs. at each end, and by an unlike parallel force of 5 lbs. at a point 1 foot from one end; find the force necessary to maintain equilibrium.

9. A man can just lift 112 lbs.; a rod 3 ft. long which can turn about one end has a weight fastened at a distance of 1 ft. from that end; what is the greatest magnitude of the weight if the man is to lift it by applying a lifting force to the other end of the rod?

10. A horizontal rod 15 ft. long can turn about one extremity which is fixed; a force of 10 lbs. acts upwards at the other end and one of 20 lbs. is applied downwards at a point between; find where the 20 lbs. is applied.

11. Three like parallel forces of 5 lbs., 7 lbs. and 9 lbs. act in lines whose distances apart are 3 ft. and 4 ft.; find their resultant.

12. Two like parallel forces of 5 lbs. and 9 lbs. and an unlike parallel force of 7 lbs. are such that the line of action of the 7 lbs. is between the other two and distant 3 ft. from the former and 4 ft. from the latter force· find their resultant.

13. Two men, one stronger than the other, have to remove a block of stone weighing 300 lbs. with a light pole whose length is 6 ft.; the weaker man cannot carry more than 100 lbs.; where must the stone be fastened to the pole so as just to allow him his full share of the weight?

14. Two men, one stronger than the other, have to remove a block of stone weighing 270 lbs. with a light pole whose length is 12 ft.; the stronger man is just able to carry 180 lbs.; how must the stone be suspended from the pole so as to allow him his maximum weight?

15. A light rigid bar 30 ft. long has suspended from its middle point a weight of 700 lbs., and rests on two walls 24 ft. apart, so that 1 foot of it projects over one of them; a weight of 192 lbs. is suspended from a point 2 ft. from the other end; what is the pressure borne by each of the walls?

16. A light plank 20 ft. long rests on the top of a wall; at one end is a man weighing 12½ stone, at the other a boy weighing 6½ stone, and 2 ft. from that a basket of eggs weighing 7 lbs.; how much of the plank is on each side of the wall?

17. The resultant of two unlike parallel forces is 2 pounds and acts at distances 6 in. and 8 in. from them; find the forces.

18. A light rigid rod 10 ft. long can turn freely about a point 4 ft. from one end, at which end a weight of 210 lbs. is hung; if there is also a weight of 140 lbs. suspended at the middle point of the rod, what weight must be suspended at the other end to maintain equilibrium?

19. A light rigid rod 12 ft. long turns freely about a point 9 ft. from one end at which a weight of 100 lbs. is suspended; at the middle point is suspended a weight of 50 lbs.; what weight must be suspended from the other end to maintain equilibrium?

20. A light rigid rod 20 ft. long is supported in a horizontal position on two posts 9 ft. apart, one post is 4 ft. from the end of the rod; from the middle point of the rod a weight of 63 lbs. is suspended: find the pressures on the posts.

21. Unlike parallel forces of 3 lbs. and 7 lbs. act at points of a bar 10 ft. apart; find the least length of the bar that it may be capable of being kept in equilibrium by a single force acting upon it.

22. A rod 3 ft. long is suspended by two vertical strings one attached to each end of the rod; two equal weights are suspended from the rod at points distant 9 in. and 21 in. respectively from one end of the rod; find the greatest possible magnitude of the equal weights in order that neither of the forces exerted by the strings may exceed 1 cwt.

23. If in Question **22** the rod is 2 ft. long and the distances of the points of suspension of the weights 4 in. and 16 in. respectively from one end of the rod; find how great the equal weight may be if the strings will break under any force exceeding 1 cwt.

24. Six like parallel forces of 1 lb., 2 lbs., 3 lbs., 4 lbs., 5 lbs., 6 lbs. respectively are applied to a rigid rod at points one inch apart; find their resultant.

25. Six like parallel forces of 7 lbs., 6 lbs., 5 lbs., 4 lbs., 3 lbs., 2 lbs. are applied to a rigid rod at points 1 foot apart; find their resultant.

26. Weights of 3 lbs., 5 lbs., 7 lbs. and 9 lbs. are suspended from a light rigid rod 8 ft. long at points equally distant from each other; find where a force must be applied to the rod to support it.

27. Weights of 2, 4, 6, and 8 lbs. are suspended from a light rigid rod 12 ft. long at points equally distant from each other; find where the rod must be supported that it may be in equilibrium when horizontal.

28. A light horizontal rigid rod 3 ft. long has a weight of 15 cwt. suspended from a point on it, and it is supported by four strings, which apply forces to it which are in the ratio of 1 : 2 : 4 : 8, and which are fastened to the rod at points each 1 foot apart; where is the weight fastened, and what force does each string apply to the rod?

CHAPTER XII.

THE CENTRE OF PARALLEL FORCES.

160. In the preceding chapters we have learned how to determine not only the magnitude of the resultant of a given system of parallel forces and the sense in which it acts, but also its position relatively to the forces of the system. This position is determined when we know any single point in the line of action of the resultant. The resultant itself is parallel to the forces and its line of action passes through every such point.

Now suppose the forces of such a system (still remaining parallel to each other) to be turned each about its point of application; it is the purpose of this chapter to prove that their resultant (still remaining parallel to the system) will also turn about a certain fixed point in its line of action.

This fixed point is called **the Centre of the Parallel Forces.**

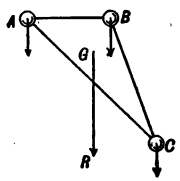


Fig. i.

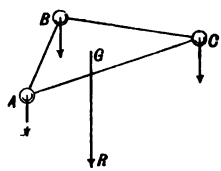


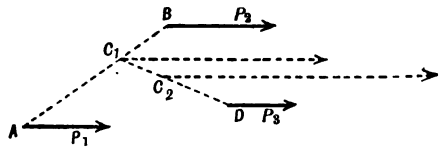
Fig. ii.

For example, let ABC (Fig. i.) be three heavy iron balls fixed at the corners of a light frame of rigid rods.

The weights of these balls are three parallel downward forces, and their resultant R is a vertical downward force equal to the sum of the three weights, whose position relatively to ABC may be found by Art. 156. If now the system of bodies be moved from the position shewn in Fig. i. to that shewn in Fig. ii., the directions of the three original forces will be changed as regards the relative positions of the balls, but it will be found that there is a point G, whose position relative to A, B, C is *fixed*, through which R *must pass*.

161. PROP. *To prove the existence of a Centre of Parallel Forces.*

Let P_1, P_2 be the given magnitudes of two parallel forces applied to a rigid body at given points A and B.



Their resultant is parallel to P_1 and P_2, and is equal in magnitude to $P_1 + P_2$, and its line of action cuts AB in a point C_1, such that $C_1A \times P_1 + C_1B \times P_2 = 0$.

This point C_1 will be the same for all *directions* of the given parallel forces provided their points of application are unchanged.

Hence, the parallel forces P_1, P_2 may be replaced by a single parallel force whose magnitude is $P_1 + P_2$ applied at the *fixed* point C_1.

Let P_3 be a third parallel force and let it be applied to the rigid body at the given fixed point D.

Then we have two forces $(P_1 + P_2)$ acting at C_1, and P_3 acting at D.

As before, their resultant is parallel and equal to

$$(P_1 + P_2) + P_3,$$

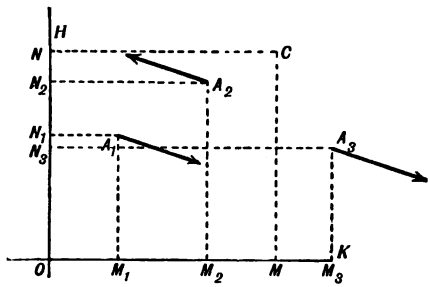
and its line of action cuts C_1D in a fixed point C_2, such that $C_2C_1 \times (P_1 + P_2) + C_2D \times P_2 = 0$.

Hence the *three* parallel forces P_1, P_2, P_3 may be replaced by a single parallel force whose magnitude is $(P_1 + P_2 + P_3)$ applied at the *fixed* point C_2; and so on.

162. PROP. *To find the* Centre *of a given system of Parallel Forces.*

Let A_1, A_2, A_3... be the given fixed points at which the given parallel forces P_1, P_2, P_3... are applied to a rigid body in one plane.

In the plane of the forces take any two lines *OH*, *OK* mutually at right angles.



Draw A_1M_1, A_2M_2... perpendicular to *OK* and draw A_1N_1, A_2N_2... perpendicular to *OH*.

The resultant of the system is a force of magnitude $(P_1+P_2+P_3+\ldots)$ parallel to the system and it always passes through the *Centre* of the system.

Let *C* be the centre of the system. Draw *CM*, *CN* perpendicular to *OK* and *OH* respectively.

Now by Art. 161 the position of *C* depends on the positions of A_1, A_2, A_3,... and on the *magnitudes* of P_1, P_2, P_3,... and it does *not* depend on the direction of the forces P_1, P_2, P_3...; hence

We may first suppose the forces to be all parallel to the line *OH*.

Then since the force $-(P_1+P_2+P_3+\ldots)$ acting through *C* is in equilibrium with the forces P_1, P_2, P_3... acting at A_1, A_2, A_3..., we have, by taking moments about *O* (remembering that all the forces are parallel to *OH*),

$$-(P_1+P_2+P_3+\ldots)\times OM+P_1\times OM_1+P_2\times OM_2+P_3\times OM_3+\ldots = 0;$$

$$\therefore\quad OM=\frac{P_1\times OM_1+P_2\times OM_2+P_3\times OM_3+\ldots}{P_1+P_2+P_3+\ldots}$$

$$=\frac{\Sigma[P_1\times OM_1]}{\Sigma[P_1]}.$$

Next the position of the Centre will not be changed if we suppose the system of forces to be parallel to OK;

Hence similarly, assuming that the forces are parallel to OK, we have

$$ON = \frac{P_1 \times ON_1 + P_2 \times ON_2 + P_3 \times ON_3 + \dots}{P_1 + P_2 + P_3 + \dots}$$

$$= \frac{\Sigma[P_1 \times ON_1]\dagger}{\Sigma[P_1]}.$$

Thus the position of C is found.

163. NOTE. The student must notice carefully that the position of the centre of a system of *parallel* forces depends (i) on their points of application, (ii) on their relative magnitudes; it does not depend on the *direction* in which they act.

164. Moreover since it depends on their *relative* magnitudes, it follows that if each force were doubled or tripled... the position of the Centre of Parallel Forces would not be changed.

This also appears from the formula

$$OM = \frac{\Sigma[P_1 \times OM_1]}{\Sigma[P_1]}.$$

For if the forces P_1, P_2, P_3... are each multiplied by the same number the value of OM is unchanged.

Again, it appears from the *symmetry* of the formula

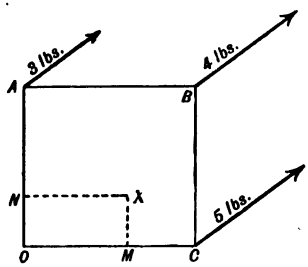
$$OM = \frac{\Sigma[P_1 \times OM_1]}{\Sigma[P_1]},$$

that in whatever *order* we take the forces, the value of OM is the same; so that the formula can give only *one* position for the Centre.

† $\Sigma[P_1 \times ON_1]$ is an abbreviation for the sum of the products

$$P_1 \times ON_1 + P_2 \times ON_2 + P_3 \times ON_3 + \dots$$

Example. Three parallel forces of 3 lbs., 4 lbs., 5 lbs. *act at the angular points* A, B, C *of a square* ABCO *whose side is* 4 in; *find* the Centre of these parallel forces.



Let X be the required centre; draw XM, XN perpendicular to OC, OA respectively.

Then by Art. 162

$$ON = \frac{3 \times 4 \text{ in.} + 4 \times 4 \text{ in.} + 5 \times 0 \text{ in.}}{3+4+5}$$

$$= \frac{28}{12} \text{ in.} = \underline{2\tfrac{1}{3} \text{ in.}}$$

$$OM = \frac{3 \times 0 \text{ in.} + 4 \times 4 \text{ in.} + 5 \times 4 \text{ in.}}{3+4+5}$$

$$= \frac{36}{12} \text{ in.} = \underline{3 \text{ in.}}$$

EXAMPLES. XXXIV.

Find the centre of the ten following systems of parallel forces:

1. Forces of 4 lbs., 5 lbs., 6 lbs. acting at A, B, C in the square above.

2. Forces of 6, 7, 8, 9 lbs. acting at A, B, C, O as above.

3. Forces of 10, 10, 12 lbs. acting at B, C, O of a square $ABCO$, where $AB = 1{\cdot}6$ inches.

4. Forces of 10, 10, 20 lbs. acting at A, B, C in an equilateral triangle ABC, where $AB = 10$ inches.

5. Four equal like parallel forces acting at the angular points of a given parallelogram.

6. Three equal like parallel forces acting at the angular points of a given triangle.

7. Three like parallel forces acting at the three angular points of a given triangle, proportional respectively to the opposite sides.

8. Three equal parallel forces (not all like) acting at the three angular points of a given triangle.

9. Three parallel forces (not all like) acting at the three angular points of a given triangle, proportional respectively to the opposite sides.

10. Four equal like parallel forces acting at the angular points of a given quadrilateral.

---

11. Use the result of Question **10** to prove that the lines joining the middle points of opposite sides of a quadrilateral bisect each other.

12. *ABCD* is a parallelogram; like parallel forces proportional to 6, 10, 14, 10 act at *A*, *B*, *C*, *D* respectively; shew that the centre and resultant of the parallel forces will remain unchanged if, instead of these forces, the parallel forces 8, 12, 16, 4 act at the points of bisection of the sides *AB*, *BC*, *CD*, *DA* respectively.

13. *ABCD* is a square whose side is 17 inches and *E* the intersection of its diagonals; like forces proportional to 3, 8, 7, 6 and 10 act at the points *A*, *B*, *C*, *D*, *E* respectively; prove that the distances of their centre from *AB* and *AD* are 9 inches and 10 inches respectively.

CHAPTER XIII.

CENTRE OF GRAVITY.

165. We have said that *force* is applied to mass. There are **three ways** in which force may be applied to **mass**.

I. As a **Pressure**.

II. As a **Tension**.

III. As an **Attraction** or *repulsion*.

166. A force is called a **pressure** when it is so applied that it causes the external particles of the mass to tend to compress the neighbouring particles of the mass closer together.

Suppose, for example, we take a piece of stick, such as a penholder, and press with two of our fingers one on each end of the stick in opposite directions. Then we are applying a force at each end of the stick; and these forces tend to compress the particles of the stick closer together. The particles *are* pressed closer together in consequence of the action of the two forces and the particles of the stick are in a state of *strain*. The amount of compression is however too small to be observed.

167. A force is called a **tension** when it is so applied to a mass that it causes the external particles of the mass to tend to *extend* the neighbouring particles of the mass so as to occupy a larger space.

Suppose, for example, we take a piece of stick, or of thread, and taking hold of each end with a finger and thumb, pull on the stick (or thread) in opposite directions with each hand. Then we are applying a force at each end of the stick; and these forces tend to pull the particles of the stick further apart. The particles *are* extended more widely in consequence of the action of the two forces; and the particles of the stick are in a state of *strain*. The amount of extension is however too small to be observed.

168. A force is called an **attraction** when there is a force acting upon each particle of the mass, which is quite independent of the internal actions and reactions of the neighbouring particles of the mass.

Such a force is said to act *from a distance* because it is not communicated to each particle by the action of neighbouring particles.

Such a force is the force called **weight**. Every particle of matter on the surface of the earth is attracted by the earth in such a way that each of its particles is pulled towards the centre of the earth by a force which is proportional to its mass and inversely proportional to the square of its distance from the centre of the earth.

Thus if the distance of a particle from the earth's centre is changed, its weight is changed. And since the shape of the earth is not exactly a sphere it happens that the weight of the same particle is different at different places on the earth's surface. See p. 31.

169. It can be shewn that the weight of a given mass at the same point of the earth's surface is always the same. Hence we are justified in taking the weight of a pound at Greenwich as our unit Force in Statics.

A certain pressure (or a certain tension) is said to be *equal* to the unit force [Art. 32] when it is such that if applied vertically to the mass 1 lb. under the action of *gravity* (that is, the earth's attraction) it will keep it at rest.

NOTE. A *pressure*, a *tension*, a *weight* are not each a different *kind of force*; each is a force [DEF. Art. 31], but the manner of their application to mass is different.

THE WEIGHT OF RIGID BODIES CONSIDERED AS SYSTEMS OF PARTICLES.

170. The centre of the earth is at so great a distance from its surface when compared with the greatest linear dimension of any mass upon which we can make any experiment that we may consider the weights of the particles of any mass to be **parallel vertical** forces, each force being proportional to the mass of the particle.

The distance of the centre of the earth is about 4000 miles. A mass whose greatest dimension is a few yards is perhaps as large a mass as it is possible to consider or treat as a rigid body; hence in considering the weights of the particles of a body to be *parallel* forces we are at most neglecting an angle AOB when AB = say 10 yards and $OA = 4000 \times 1760$ yards, that is we are neglecting an angle of less than $\frac{3}{10}$ of a second. An angle of $\frac{3}{10}$ of a second is as small an angle as can possibly be observed with accuracy, even under the most favourable conditions.

171. Consider now two particles A and B rigidly connected by a light straight rod.

The weights of these particles are two parallel vertical forces acting the one at A the other at B.

These two forces are statically equivalent to a single force equal in magnitude to their sum, parallel and of like sense, acting at a point C between A and B such that $AC \times$ the force at $A = BC \times$ the force at B. [Art. 156.]

Hence the two particles A and B when under the action of gravity may be kept in equilibrium by an upward vertical force acting at C whose magnitude is equal to the sum of the weights of the particles A and B.

The point C is the centre of the parallel forces A and B, namely, the weight of the two particles.

172. Suppose then we fasten a light string to the rod at C and apply this upward vertical force to the rod, then the two particles will be in equilibrium under the action of this force and of their own weight, no matter what angle the line AB makes with the string.

Hence the weights of the two particles A and B are statically equivalent to the weight of a single particle at C whose weight is equal to the sum of the weights of A and B.

The point C is called the **Centre of Gravity** of the particles A and B.

173. DEF. The **Centre of Gravity** of a system of particles is the centre of the system of parallel forces which consists of the weights of the particles.

174. When the system of particles is in the form of a rigid body the Centre of Gravity is a fixed point in or near the body; and a downward force applied at that fixed point is statically equivalent to the weights of all the particles for all positions of the body, hence

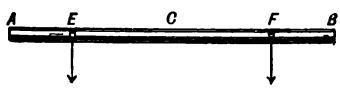
175. The Centre of Gravity of a *rigid body* is a point such that when that point is supported, the body will be in equilibrium under the action of gravity in *whatever position* the body may be placed.

The Centre of Gravity in many cases is not within the substance of the rigid body. It must of course in any case be rigidly connected with the body for the purpose of the above statement.

The student is recommended to take some rigid body whose centre of gravity is not *within* its substance, say a Windsor chair, (or a box without a lid,) and find G its centre of gravity, rigidly connecting it with the chair by a system of fine wires, so that the chair can be supported by another wire fastened to the wires meeting at G. It will be found that in whatever position the chair is placed when suspended from G, it will rest without any tendency to move.

176. A **uniform** rod is a straight rod made of material of uniform density and whose breadth and thickness are the same at every point of its length.

177. PROP. *The Centre of Gravity of a* uniform rod *is at its middle point.*



Let AB be a uniform rod, let C be its middle point.

Since the rod is uniform the rod is symmetrical about C; that is, corresponding to any particle E of the rod between C and A there is an exactly equal particle F between C and B such that $CE = CF$.

The centre of gravity of every pair of such particles is at C.

Hence the centre of gravity of all the particles of the rod is at C.

178. Hence the weight of an uniform rod is statically equivalent to a single force acting vertically downwards at its middle point C, whose magnitude is equal to the sum of the weights of all the particles of the rod.

Example. A uniform rod AB 3 ft. *long weighs* 3 lbs.; *at the point* A *a force of* 3 lbs. *is applied to the rod in a vertical direction downwards; shew how to apply a force to the rod which will keep it in equilibrium.*

Let C be the middle point of the rod. The rod being uniform its weight is equivalent to a force of 3 lbs. acting vertically downwards at C.

Hence we have to find a force and its point of application such that it is in equilibrium with the two parallel forces of 3 lbs. each, acting at A and C respectively.

The required force is the anti-resultant of the two like parallel forces of 3 lbs. each. It is therefore a force of 6 lbs. acting vertically upwards at the middle point of AC.

179. The forces which occur in *nature* are applied to bodies either at a portion of their *surface* or at a part of their *volume* (as in the case of weight).

Consider a pressure applied to a portion of the surface of a body, such as the cube in the figure on the next page. We may look upon it as a solid bundle of minute forces whose lines of action form a cylinder. Suppose these forces all symmetrical about a certain plane, say the plane $ELCO$. Then if we select any minute force on one side of the plane, there is an exactly symmetrical equal force on the other side of the plane, and their resultant will act *in the plane.*

Hence, the whole bundle of forces will be equivalent to a series of forces acting in the plane about which they are symmetrical. This system of forces will have a resultant acting in this plane; and it is this *resultant* of a bundle of forces which is the *force* of Statics and which has *line* of action and *point* of application.

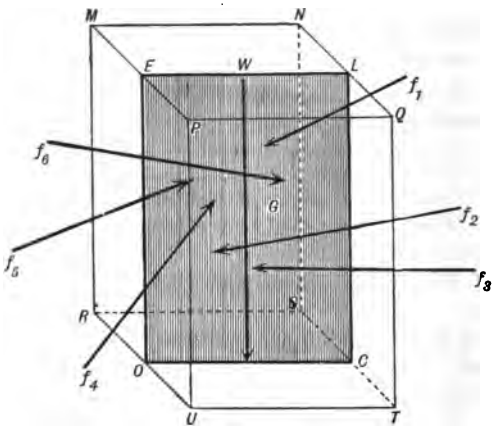
Such a force as the theoretical **linear force** of statics **does not exist in nature**, but it is a convenient *representation* of the cylindrical or conical or solid force of nature.

180. In practical problems the various forces (viz. pressures, tensions, weights) which act on a body are in very many cases all *symmetrical about the same plane;* in such cases the corresponding theoretical linear forces are all in the same plane.

These symmetrical problems form the very numerous class of problems which can be solved by the consideration of *forces in one plane.*

Illustration.—Suppose *MPQN, RUTS* to represent a cubical block of stone; suppose it to rest on a horizontal plane. The forces acting on it are:

First, its *weight;* this force is applied to each particle of the stone; we therefore consider the weight to be a multitude of very small forces all parallel and symmetrical about the plane *EOCL*; for to each particle on one side of this plane there is an equal particle similarly placed on the other side of this plane. Therefore the weight considered as the resultant of this multitude of parallel forces, acts in the plane *EOCL.*



Next, the *pressure* of the plane on *RUTS*, the base of the stone. This is applied to the surface of the stone, and is one part of the stress set up between the stone and the plane. If the plane and the stone be perfect planes the pressure will be *uniformly* distributed over the *surface.* Thus we have a multitude of vertical forces upwards, forming, if we represent their magnitudes by finite straight lines, a portion of a solid vertical cylinder. It is the theoretical resultant of this solid cylinder of force which is the linear force of Statics called the pressure on the plane.

Next, suppose we apply *a force* f_1 to the stone in any direction. This force would practically be applied to part of the *surface* of the cube, but if applied symmetrically about the plane, may be represented by a linear force *in the plane OCLE.*

And so on, for other forces applied to the cube, whether by means of rods or strings or whether caused by the pressure of other solid bodies.

EXAMPLES. XXXV.

1. A uniform rod weighing 10 lbs. has a weight of 10 lbs. fastened to one extremity; find the centre of gravity of the two.

2. A uniform rod weighing 5 lbs. has weights of 3 lbs. and 5 lbs. fastened to its extremities; where must an upward force be applied to the rod which will support it?

3. A ladder 22 ft. long and weighing 44 lbs. has its centre of gravity 10 ft. from one end; if it is carried by two men lifting it at each end, what weight do they each lift?

4. A pole 40 ft. long weighing 20 lbs. has its centre of gravity 16 ft. from one end; it is carried by two men, one of whom lifts it at its heavier end; where must the other lift it that he may support an equal weight with the first man?

5. A ladder 50 ft. long and weighing 100 lbs. is carried by two men, one lifts it at one end and the other at a point 2 ft. from the other end; the first carries two-thirds of the weight which the second carries: where is the centre of gravity of the ladder?

6. A pole 8 ft. long weighing 10 lbs. has weights of 8 lbs. and 4 lbs. fastened one to each of its ends; the centre of gravity of the whole is at the middle of the rod: where is the centre of gravity of the pole?

7. A pole 10 ft. long weighing 20 lbs. has a weight of 12 lbs. fastened to one end; the centre of gravity of the whole is 4 ft. from that end: where is the centre of gravity of the pole?

8. A pole weighing 20 lbs. is 12 ft. long, and its centre of gravity is 4 ft. from one end; if the pole is supported at its middle point, find where a weight of 10 lbs. must be fastened to it that it may be in equilibrium.

9. A pole 6 ft. long is found to balance about a point 2 ft. from one end; when a weight of 1 lb. is fastened to that end, and a weight of 3 lbs. is fastened to the other end, the pole is found to balance about its middle point: find the weight of the pole.

10. A ladder 20 ft. long has its centre of gravity at a point 8 ft. from one end; when a weight of 10 lbs. is fastened to that end and a weight of 20 lbs. to the other end, it is found to balance about a point 8 ft. from this other end: find the weight of the ladder.

11. Four weights of 7 lbs., 1 lb., 3 lbs. and 5 lbs. respectively are placed a foot apart on a uniform rod 3 ft. long weighing 8 lbs.; find the point on which the rod will balance.

12. Four weights 1 lb., 4 lbs., 5 lbs. and 3 lbs. respectively are placed 2 ft. apart on a rod 6 ft. long weighing 3 lbs. whose centre of gravity is 2 ft. from the end at which is the 1 lb. weight: find the centre of gravity of the whole.

13. Weights of 3 lbs., 5 lbs., 7 lbs., 9 lbs. are fastened to a rod 6 ft. long at intervals of 2 ft.; the rod weighs 24 lbs. and the centre of gravity of the whole system is at the middle point of the rod; where is the centre of gravity of the rod?

14. A uniform bar is 18 inches long, weighs 3 lbs. and can turn about a fixed point 3 in. from one extremity; what weight must be fastened to this extremity that the bar may be in equilibrium, when a weight of 2 lbs. is fastened to the other extremity?

15. A uniform rod OA 12 inches long is suspended by two vertical strings attached to the rod at O and A; weights of 2 lbs. and 7 lbs. are fastened to the rod at points distant 1 inch from O and 2 inches from A respectively; if the strings break when subjected to a strain of more than 7 lbs., find the greatest weight the rod can have without breaking either of the strings.

16. A uniform rod 2 ft. long is suspended by two vertical strings attached to the ends of the rod; weights of 7 lbs. and 1 lb. are attached to the rod at points distant 4 in. from the ends. If the strings break if subjected to a strain of more than 10 lbs. find the greatest weight the rod can have.

17. A uniform beam weighing 700 lbs. and 30 ft. long rests on two walls 24 ft. apart so that 1 foot of it projects over one of them; a man weighing 192 lbs. stands 2 ft. from the other end: find the pressure on each of the walls.

18. A uniform plank 20 ft. long weighing 14 lbs. rests on the top of a wall; at one end is a man weighing 168 lbs., at the other a boy weighing 98 lbs., and two feet from that end a weight of 7 lbs., the whole balances: how much of the plank is on each side of the wall?

19. Two weights of 12 lbs. and 2 lbs. hanging from the extremities of a uniform rod 3 ft. long which can turn about a fixed point, keep it at rest; if each weight is increased by 1 lb. the fixed point must be moved three-eighths of an inch nearer the less weight: find the weight of the rod.

20. A uniform lever 15 ft. long has its fulcrum at one end; a force of 30 lbs. acts upwards at the other end, and one of 20 lbs. is applied downwards at a point between; the lever is then in equilibrium; if the lever were without weight it would be in equilibrium if the points of application of the two forces were interchanged: what is its weight?

21. A uniform bar $7\frac{1}{2}$ ft. long weighing 17 lbs. rests on a horizontal table with one end projecting $2\frac{1}{2}$ ft. over the edge; find the greatest weight that can be hung at this end without making the beam topple over.

22. A uniform bar 8 ft. long rests with one end on a horizontal table; a weight of 10 lbs. is placed on that end; it is found that when the bar projects $6\frac{1}{2}$ ft. over the edge it is on the point of toppling over: what is the weight of the bar?

23. A uniform bar weighing 20 lbs. rests with one end on a horizontal table; a weight of 6 lbs. is placed on that end and a weight of 4 lbs. is placed on the end which projects over the edge; when the bar projects 8 ft. over the edge it is on the point of toppling over: what is its length?

CHAPTER XIV.

CENTRE OF GRAVITY OF A LAMINA.

181. A material lamina has thickness.

Consider, however, a geometrical plane; and suppose that on each side of this plane, material of some kind (iron, or wood, or cardboard...) is symmetrically arranged.

(Forming a sheet of iron, or a board, or a card ....)

Then the centre of gravity of the body so formed must lie in the geometrical plane about which it is symmetrical.

This follows as in Art. 180.

When the thickness of the substance on each side of the geometrical plane is small, then we have a body which we call a *lamina.*

When the substance is spread uniformly over the plane then we have a *uniform* lamina.

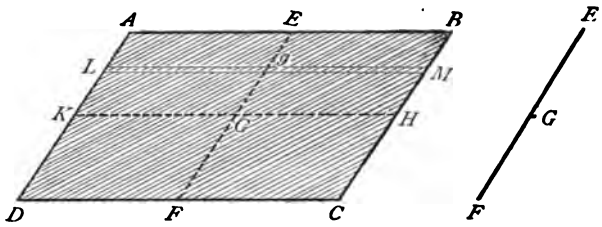
When we speak of the centre of gravity of a *triangle*, a *circle*, a *quadrilateral*, we mean the centre of gravity of a uniform lamina whose boundary is a triangle, a circle, or a quadrilateral, etc.

182. When a uniform lamina is symmetrical about a *point*, that point must be its Centre of Gravity.

For example; the C.G. of a Circle is its centre. The C.G. of a rectangle is its middle point.

183. If the substance is not uniformly spread over the plane, yet if it is spread *symmetrically*, the centre of gravity of the body so formed must lie in the plane under consideration; in such a case we have a lamina which is not uniform.

184. PROP. *To find the* Centre of Gravity *of a lamina in the form of a* **parallelogram.**



Let $ABCD$ represent the surface of a uniform lamina in the form of a parallelogram.

Suppose the lamina to be divided into a very large number of *rods*, by lines drawn parallel to the side AB.

Let LM represent one of these rods.

The C. G. of the rod LM is at its middle point g.

The C. G. of each of the other rods is at its middle point, and these middle points lie in the straight line EF which bisects AB and DC.

Therefore the weight of all the parallel rods which make up the lamina may (for the purpose of finding the C. G. of the lamina) be considered to be condensed into the straight rod EF.

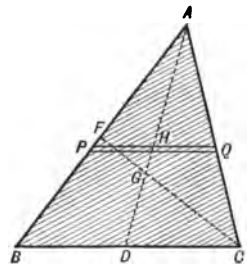
The C. G. of a *straight* rod is at some point in that rod.

Therefore the C. G. of the lamina is at some point in the straight line EF.

Similarly by dividing the lamina into rods parallel to AD, we may shew that the C. G. of the lamina lies in the straight line KH which bisects AD and BC.

Therefore the C. G. of the lamina must be at G, the point common to the straight lines EF and HK.

185. PROP. *To find the* Centre of Gravity *of a uniform* triangular *lamina.*



Consider the lamina ABC to be made up of a series of very thin parallel rods such as PQ, each parallel to one of the sides BC of the triangle, and formed by sections of the triangle made parallel to the side BC.

Since the lamina is uniform, each rod will be uniform.

Therefore the C.G. of each rod will be at its middle point. [Art. 177.]

Hence, the weight of each rod may be considered to be condensed at its middle point.

Now bisect BC in D and join AD cutting PQ in F; then AD bisects PQ in H.

For, by the similar triangles ADC, AHQ

$$DC : HQ = DA : HA,$$

and by the similar triangles ABD, APH

$$BD : PH = DA : HA;$$

therefore $$DC : HQ = BD : PH.$$

But $$BD = DC, \therefore HQ = PH.$$

Therefore the C.G. of each rod making up the lamina lies in the *straight* line AD.

But the Centre of a series of parallel forces whose points of application are in a *straight* line, lies also in that straight line [Art. 162].

Therefore the C.G. of the whole lamina is at some point on the line AD.

In a similar manner it can be shewn that the C.G. of the lamina is at some point on the line joining the point C to F, the middle point of the line AB.

Therefore the Centre of Gravity of the lamina must be at G, the point of intersection of the lines AD, CF.

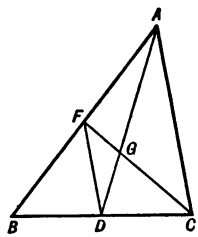
186. PROP. *To prove that the* C.G. *of a uniform triangular lamina cuts the line joining an angular point to the middle point of the base, in the ratio* 2 *to* 1.

Let ABC be the triangle.

Bisect BC, AB in D and F;

join FD;

join AD, CF intersecting in G.



Then, by Art. 185, G is the C.G. of the lamina. Now, since

$$AF : FB = CD : DB,$$

$\therefore$ FD is parallel to AC,

and $\therefore$ ACB, FDB are similar triangles;

$$\therefore AC : FD = AB : FB = 2 : 1.$$

Also, since AGC, DGF are similar triangles,

$$AG : GD = AC : FD$$
$$= 2 : 1;$$
$$\therefore AG = \text{twice } GD,$$
$$\text{and } AD = \text{thrice } GD.$$

Q.E.D.

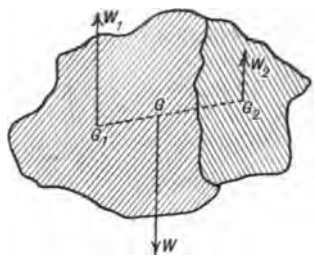
NOTE. The C.G. of a triangular lamina coincides with that of three equal particles placed one at each of its angular points.

For the C.G. of two equal particles of weight W placed at B and C respectively, is at D; we may therefore replace these two particles by a particle of weight $2W$ at D.

The C.G. of $2W$ at D and W at A is at G where $AG = 2GD$.

NOTE. The C.G. of a parallelogram is at the point of intersection of its diagonals. This may be deduced from Art. 185 or from Art. 184.

187. PROP. *Having given the weight and* C.G. *of a body, and also the weight and* C.G. *of a portion of the same body, to find the* C.G. *of the remaining portion.*



Let G be the C.G. of the given body and W its weight.

Let G_1 be the C.G. of the given portion and W_1 its weight.

If G_2 be the required C.G. of the remainder and W_2 its weight, then $W_2 = W - W_1$.

Also G is the centre of the two parallel forces W_1, W_2 applied at the points G_1, G_2;

$\therefore$ G_1, G, G_2 are points in a straight line,

and $GG_2 \times W_2 = G_1G \times W_1$;

$$\therefore GG_2 = \frac{W_1}{W - W_1} \times G_1G.$$

Hence G_2 is found by producing G_1G a distance given by the above equation.

Example. A piece of cardboard is in the shape of the parallelogram ABCD; EGF *and* HGK *bisect the sides* AB, DC *and* BC, AD *respectively; if the parallelogram* GHCF *be cut away find the* C. G. *of the remainder.*

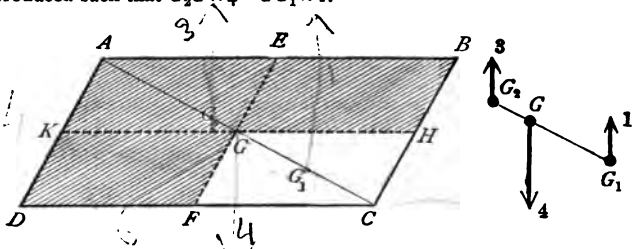
Join AGC; let G_1 be the C. G. of the parallelogram $GHCF$.

Then we have to find the C. G. of a part of a body whose C. G. is G, the C. G. of the part taken away being G_1.

Let the weight of the parallelogram $GHCF$ be unity.

Then we have to find the line of action of the resultant of two unlike parallel forces; one of 4 units acting downwards at G and the other 1 unit acting upwards at G_1.

The resultant is 3 units acting at some point G_2 in the line G_1G produced such that $G_2G \times 4 = GG_1 \times 1$.



Hence if we divide G_1G into three equal parts and produce it to G_2 a distance equal to one of the parts we shall find the C. G. of the remainder of the cardboard.

188. *A heavy body is suspended from a fixed body by a string; prove that the direction of the string must be vertical and must pass through the Centre of Gravity of the body.*

The weight of the body is equivalent to a single vertical force acting vertically downwards and applied to the body at its centre of gravity.

The only other force acting on the body is the tension of the string.

Therefore these two forces, the weight and the tension, must be equal, opposite, and in the same line of action.

But the tension acts in the direction of the string.

Therefore the direction of the string must be in the same line as that in which the weight acts. Q. E. D.

189. To find the Centre of Gravity of a number of heavy bodies whose weight is known, and the positions of whose Centres of Gravity are given, we proceed as in Art. 162 or as in Art. 161.

For the weight of a heavy body is a force acting in a fixed direction, viz. vertically downwards, and it is applied at the Centre of Gravity of the body.

Thus the problem is the same as that of finding the position of the centre of a given system of like Parallel Forces having given points of application.

EXAMPLES. XXXVI.

1. A cross is made up of six equal squares; find its centre of gravity.

2. Weights of 1 lb. are placed at each of three corners of a square and a weight of 2 lbs. at the fourth corner; find the C.G. of the four weights.

3. Weights of 2 lbs. are placed at each of three corners of a square and a weight of 1 lb. at the fourth corner; find their C.G.

4. If the triangle formed by joining the middle points of the sides of a triangular lamina be removed, prove that the C.G. of the remainder coincides with that of the original triangle.

5. Two rectangular pieces of cardboard of lengths 6 and 8 inches and breadths 2 and $2\frac{1}{2}$ in. respectively are placed touching but not overlapping one another on a table to form a T-shaped figure, the former piece forming the cross bar. Find the position of the centre of gravity.

6. From the corner of a square piece of cardboard whose side is 6 inches another square whose side is 2 inches is cut away; find the C.G. of the remaining piece.

7. $ABCD$ are the corners of a square piece of cardboard; AC, BD meet in E; if the triangle AEB be cut away, find the C.G. of the remainder.

8. A quarter of a triangle is cut off by a line drawn parallel to one of its sides bisecting each of the other sides, find the C. G. of the remainder.

9. A circular plate of wood has a circular hole cut in it whose diameter is half that of the plate and whose centre divides a diameter of the plate in the ratio 2 : 3; find the position of its centre of gravity.

10. A circular board has two equal circular holes cut in it, the centre of these holes being at the middle points of two radii at right angles to each other; the radius of each hole is one-third that of the board; find the centre of gravity of the remainder.

11. Two equal rods AB, BC are fixed upon a circular board so as to coincide with the chords of two adjacent quadrants; the weight of each rod is equal to that of the board; find the centre of gravity of the whole body.

12. The diagonals of a square plate $ABCD$ intersect in O, the triangle BOC is cut out and placed so as to fit upon the triangle AOD and the two portions are firmly connected in this position; find the centre of gravity of the body thus formed.

13. $ABCD$ is a square plate; E and F being the middle points of AB and BC; the plate is bent along EF so that the triangle EBF lies flat on the other side of the plate; find the centre of gravity in this position.

14. A square is described on the base of an isosceles triangle. What is the ratio of the altitude of the triangle to its base when the C.G. of the whole figure is at the middle point of the base?

15. If a triangle have a side upon which it will not stand upon a horizontal plane with its plane vertical, that side is the shortest and is not half the length of the longest side.

16. On a uniform triangular plate straight lines are drawn joining the middle points of the sides. Two of the triangular portions marked off between these lines and the angle of the triangle are covered with triangular plates of the same material and thickness as the given triangle; find the centre of gravity of the whole.

17. A square is described on the base of an equilateral triangle; find the C.G. of the whole figure.

18. A system of three equal particles connected by rigid wires without weight forms a triangle and when hung up by the middle point of one side rests with that side horizontal; prove that the triangle is isosceles.

19. An isosceles right-angled triangle is described on the side of a square as hypotenuse, and its vertex turned away from the square; find the C.G. of the whole figure.

20. $ABCD$ is a rectangle; A is joined to E the middle point of CD; find the C.G. of $ABCE$.

21. A triangular lamina EDC of the same weight as a square lamina $ABCD$ (E being the middle point of AB) is laid upon the square lamina in the position indicated by the letters; find the C.G. of the system.

22. If weights be placed at the angular points of a triangle, respectively proportional to the sum of the sides which meet at those points, prove that their C.G. will coincide with that of the perimeter of the triangle.

23. The C.G. of the two complements which are about the diagonal of any parallelogram is in that diagonal.

24. ABC is a triangle, D is a fixed point in BC; a triangle PBC is cut away whose vertex P is in AD; prove that whatever be the position of P the C.G. of the remainder lies on a fixed straight line.

25. If there are two triangles on the same base and between the same parallels, prove that the distance between their centres of gravity is one-third the distance between their vertices.

26. Find the centre of gravity of four equal heavy particles in one plane; thence shew that the lines bisecting pairs of opposite sides of any quadrilateral bisect each other.

27. A triangular lamina is hung up by one of its angular points and when in equilibrium the opposite side is horizontal; prove that the triangle is isosceles.

28. Shew that the centre of gravity of a plane quadrilateral does not coincide with that of four equal particles placed at its angular point except the quadrilateral be a parallelogram.

29. Shew by the method of Arts. 187, 188 that the centre of gravity of a lamina in the form of a parallelogram is at the point of intersection of the diagonals.

30. Prove the following construction for finding the C.G. of three equal weights at the points A, B, C; bisect BC in D and divide AD in G so that $AG=2GD$; G is the required C.G.

31. The C.G. of three particles placed one at each of the angular points A, B, C of a triangle such that the weight of each is proportional to the opposite side, is at the centre of the circle inscribed in the triangle.

32. Three particles placed at the angular points A, B, C of a triangle are proportional to the areas of the triangles OBC, OCA, OAB respectively, where O is the centre of the circumscribing triangle; shew that their C.G. is O.

33. The C.G. of three uniform rods forming a triangle A, B, C is at the centre of the inscribed circle of the triangle formed by joining the middle points of the sides of the triangle ABC.

CHAPTER XV.

BODIES ON A HORIZONTAL PLANE.

190. When a body stands on a plane, its **base** is the area enclosed on the plane by a string drawn tightly round all the points in which the body touches the plane.

191. PROP. *A rigid body under the action of gravity only, standing on a horizontal plane, is in equilibrium provided the vertical line through its centre of gravity cuts the plane at some point within its base.*

The proof of the above general proposition involves the consideration of parallel forces *not* all in the same plane and therefore cannot be fully treated here. Its truth depends on the following.

I. When a *rigid* body rests on a horizontal plane the plane is supposed rigid also, and therefore capable of applying to the body at *any* point of its contact the *vertical* force **upwards** that may be necessary to maintain equilibrium.

II. The line of action of the resultant of any number of **like** vertical forces applied to the rigid body cuts the horizontal plane at a point *within* the base; and by properly arranging the magnitudes of the vertical forces, the line of action of the resultant may be made to cut the plane at *any* chosen point within the base.

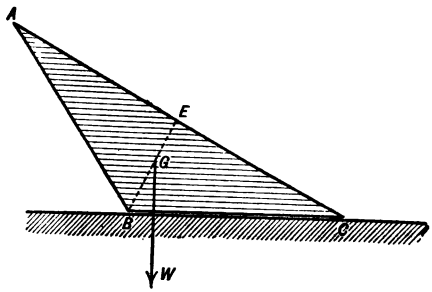
Assuming the above we can prove the proposition. For the weight of any body is equivalent to a single force downwards.

When the line of action of this vertical force downwards cuts the plane *within* the base, the upward vertical actions of the plane on the body will be so arranged that the line of action of their resultant coincides with that of the weight of the body and the magnitude of this resultant will just equal the weight of the body. Therefore the forces acting on the body will be in equilibrium.

When the line of action of the weight *cuts* the plane *outside* the base, then the upward actions of the plane on the body cannot be so arranged that the line of action of their resultant shall coincide with that of the weight of the body; in this case the forces acting on the body cannot be in equilibrium and the body will '*topple over*.'

192. When a rigid body and the forces acting upon it are symmetrical about a vertical plane, the problem of its stability may be treated as if its base were a *straight line*; viz. the straight line in which this vertical plane cuts the *base* of the rigid body. Compare Art. 180.

193. When the base of the body is a line BC then



I. The actions of the rigid plane on the rigid base are supposed to apply to the body, at points between B and C inclusive, any force that may be necessary to maintain equilibrium.

Hence we may, if necessary, consider any force whatever P to be applied to the body vertically *upwards* at B and any other force Q at C.

II. The resultant of two *like* forces P and Q at B and C is a like force $(=P+Q)$, acting upwards at some point **between** B and C [Art. 158]; moreover by properly arranging the magnitudes of P and Q this resultant may be made to act at *any point* between B and C.

Thus if the weight of the body cuts BC in any point between B and C, the actions of the plane on the body (by II.) can be, and therefore (by I.) will be, in equilibrium with the weight. But if the weight cuts the line BC *produced*, it cannot be in equilibrium with these forces whose resultant is a vertical force which cuts BC *between* B and C (by II.).

STABLE AND UNSTABLE EQUILIBRIUM.

194. When a body, which is acted on by forces which are in equilibrium, is slightly displaced from its position, one of three things must happen to it.

I. Either, the forces acting on it in its new position are such that they are *not* in equilibrium, but have a resultant which tends to *restore* the body to its original position.

In this case the original position of the body is said to be one of **stable** equilibrium for that displacement.

II. Or, the forces acting on the body in its new position are such that they are *not* in equilibrium, but have a resultant which tends to make the body *move further* from its original position.

In this case the original position of the body is said to be one of **unstable** equilibrium for that displacement.

III. Or, the forces acting on the body in its new position are in equilibrium.

In this case the original position of the body is said to be one of **neutral** equilibrium for that displacement.

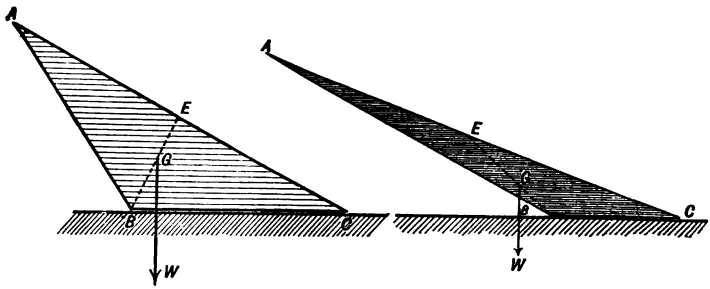
Illustration.—Take a sphere of a hard substance (a wooden ball will do), place it in a large bowl; it will come to rest at the lowest point; and if it be slightly displaced in any direction it will tend to return to its position: such a position is one of *stable* equilibrium.

Now put the same sphere on the highest point of another sphere (on the highest point of an inverted bowl); it is theoretically in a position of equilibrium; but if it be displaced ever so slightly it will tend to go further from the position of equilibrium: such a position is one of *unstable* equilibrium.

Now put the same sphere on a horizontal plane; then wherever it be placed it will be in a position of equilibrium; and if it be slightly displaced, and then left at rest, it will still be in a position of equilibrium, and will have no tendency to go further away from or to return to its original position: such a position is one of neutral equilibrium.

Example i. *An obtuse-angled triangle* ABC *is placed with its plane vertical and kept so, with its shortest side* BC *resting on a horizontal plane; find the condition that it shall not tumble over.*

Since the plane of the triangle is constrained to remain vertical the only way in which the triangle can tumble over is in its own plane.



Let B be the obtuse angle, E the middle point of AC, G the Centre of Gravity.

The vertical through the C.G. will cut BC *between* B and C unless the angle EBC is obtuse.

$\therefore$ the angle EBC must not exceed a right angle. This is the required condition.

Example ii. *A round board* ABC *weighing ten pounds is made into a table by having three equal legs (of no weight) fastened to it at right angles, at points* A, B, C *on its circumference equidistant from each other; the table is placed on a smooth horizontal plane; find the least weight which placed on the top of the table can cause it to tumble over.*

The legs of the table touch the plane in three points vertically under ABC; therefore the *base* on which the table stands is the triangle vertically under ABC.

Draw $COED$ through the centre O bisecting AB at right angles in E.

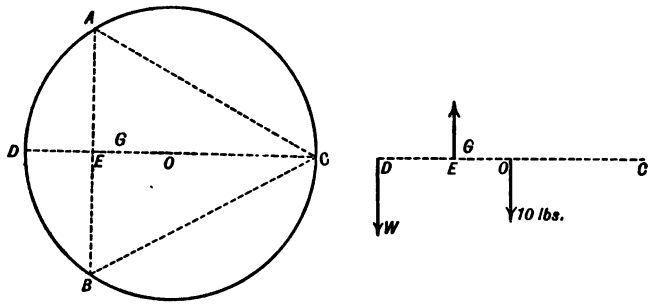
Then E is one of the nearest points in the boundary of the base to the centre of the table.

Let a weight W be placed on the table and let G be the Centre of Gravity of the table and W together; then the condition that the table should tumble over is that G shall be *outside* the triangle ABC.

The further from O we place W, the further is G from O.

$\therefore$ G is furthest from O when W is on the edge of the table.

Hence the *least* magnitude of W which will bring G on the boundary of the triangle, is when W is at D and G at E.



Now $OE=\frac{1}{2}OC=\frac{1}{2}OD=DE$.

$\therefore$ when G is at E, then $W=10$ lbs.

Hence, when W exceeds 10 lbs., the table will tumble over if W is placed at D.

EXAMPLES. XXXVII.

1. A flat board ABC in the shape of a triangle right-angled at A stands with its plane vertical and its side AC in contact with a horizontal plane; D is the middle point of AC; if the triangular portion ABD be cut away, shew that the remainder will be just on the point of falling, it being supposed that the board is constrained to remain in a vertical position.

2. A triangular lamina is placed upon a horizontal table; how far can its vertex be made to project over the side of the table when the base is kept parallel to the side of the table?

3. A parallelogram whose height is equal to its base will just stand on that base when placed vertically on a horizontal plane; find the angles of the parallelogram.

4. A uniform triangular lamina ABC is placed upon a horizontal table with the side BC on the table and parallel to the edge, and one-ninth of the area of the triangle overhangs the table. Shew that if a weight be placed at A greater than the weight of the triangle itself, the triangle will be upset.

5. A five-sided figure consisting of a square $ABCD$ with an isosceles triangle upon the side BC as base is cut out of one piece of board; find the greatest height of the triangle that the figure may stand with its side DC on a horizontal plane without tumbling over.

6. If a table stand on three legs, shew that in whatever way weights are placed upon it without upsetting it, the centre of gravity of the table and weights together will be vertically above a given triangle.

7. A circular board weighing 10 lbs. is made into a table by the addition of four legs without weight fixed perpendicular to its plane at equidistant points on its circumference; find the least weight with which it is possible to upset the table by placing the weight on it.

8. An equilateral triangular board is made into a table by the addition of three legs without weight fixed at right angles to the board at the middle points of the sides. Shew that it is possible to upset the table by putting on it a weight which is just heavier than one-third of the weight of the table.

9. A square board weighing 20 lbs. is made into a table by inserting four equal legs into it one at the middle point of each side. Three weights of 20 lbs. each are placed at three of the corners of this table as it stands on a horizontal floor. Find the greatest weight that can be placed upon the fourth corner without overturning the table, neglecting the weight of the legs.

10. A square board $ABCD$ weighing 18 lbs. is made into a table by inserting three equal legs in it, one at the corner C and the others at the middle points of the sides AB, AD. Find how great a weight may be placed upon the corner A as the table stands on a horizontal floor without overturning it, the weight of the legs not being taken into account.

11. A number of bricks, each 9 inches long, 4 inches wide, and 3 inches thick, are placed one upon another in such a way that whilst their narrowest surfaces (or thicknesses) are in the same vertical plane, each brick overlaps the other by half an inch of its length, the lowest brick resting on a horizontal plane; how many bricks may be so piled without falling?

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CHAPTER XVI.

A RIGID BODY WITH ONE POINT FIXED.

195. When a rigid body is said to have *one point* **fixed**, it is understood that there is some constraint (a hinge, or fastening of some kind), which can and does apply to the rigid body at that point whatever force is necessary to prevent the body from moving as a whole away from that point.

196. PROP. *To find the conditions for the equilibrium of a given system of forces acting upon a rigid body having one point fixed.*

If a body, which has one point C fixed, requires no additional force to keep it from moving, the given system of forces acting upon it must be in equilibrium with the force applied to the rigid body by the constraint at C.

In other words, in order that the given system of forces may be in equilibrium, their resultant must either be zero or must be a single force passing through the fixed point C.

The condition that this is so, is that **the sum of their moments about the fixed point must be zero.**

For in this case the resultant cannot be a *couple*, and it cannot be a force *not* passing through C.

Hence, any system of forces acting on a rigid body which has one point C fixed, must be in equilibrium with the force applied by the constraint at C, provided *the sum of their moments about* C *is zero.*

The above result will be found of very great importance in the theory of many machines.

For an illustration see Art. 210.

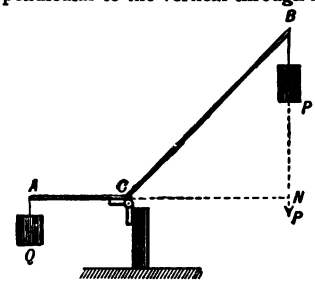
197. PROP. *A rigid body which has one point* C *fixed and is acted on by two forces, has no tendency to turn about* C *provided that the moment of one of the forces about* C *is equal and opposite to the moment of the other force about* C.

For the sum of the moments about C of the forces acting on the rigid body is zero.

The **tendency** of a given force **to turn** a rigid body about a chosen point may be measured by *the moment about that point of the given force.*

Example i. ACB *is a rigid rod without weight having a fixed point at* C; AC *is horizontal and* CB *is inclined at an angle* α *to the horizon, a* weight Q *is suspended at* A *and a* weight P *is suspended at* B. *What is the necessary condition for equilibrium?*

Draw CN perpendicular to the vertical through B.



The forces acting on ACB will be in equilibrium with the force applied by the hinge at C, provided the sum of the moments of the forces about C is zero. [Art. 197.]

The required condition is

$$AC \times Q = CN \times P = CB \cos\alpha \times P.$$

Example ii. *A heavy body of weight* W *rests on a horizontal plane, find the force in a given direction necessary to turn it over.*

Let the picture represent a plane section of the body about which it is symmetrical.

Let O be the point about which the body is to turn.

Let P be the required force, C the Centre of Gravity of the body.

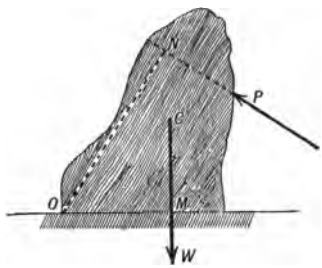
Draw ON perpendicular to the force, let the vertical through C cut the horizontal plane in M.

In such a question it is understood that the point O of the body is practically fixed, either by the roughness of the ground or by some obstacle.

When the body is on the point of turning about O, the only force acting on the body besides P and W is the reaction of the constraint at O; hence the condition that the body should be on the point of turning is that the moment about O of P = the moment about O of W; hence

$$P \times ON = W \times OM:$$

$$\therefore P = W \times \frac{OM}{ON}, \text{ the required value.}$$



Suppose the body to have turned about O through some finite angle; the perpendicular distance of W from O will be diminished; consequently the moment about O of OW will also be diminished.

Hence, the force necessary to keep the body tilted up through a finite angle if it acts along the fixed line NP is less than the above value of P.

Hence, if a force slightly greater than the P found above be applied to the body in a fixed direction the body will be tilted quite over.

EXAMPLES. XXXVIII.

In the **ten following** Examples a piece of thin cardboard is placed on a smooth table and on it is drawn a square $ABCD$ each of whose sides is 2 inches.

1. If the point A is fixed and a force of 1 lb. acts on the cardboard along the side BC, what force must act along CD to keep the cardboard at rest?

2. The point A being fixed, forces of 6 lbs. and 5 lbs. act along BC and CD; find the least force which acting at D will maintain equilibrium.

3. The point A being fixed, a force of 5 lbs. acts at B parallel to AC, and a force of 4 lbs. acts at C parallel to DB; what force acting along DB will maintain the cardboard at rest?

4. The point A is fixed; forces of 3 lbs., 4 lbs., 5 lbs. and 6 lbs. each act along AB, BC, CD, DA respectively; where must a force of 1 lb. parallel to DB be applied in AC (produced if necessary) to keep the cardboard at rest?

5. If the point E, where AC and BD intersect, is fixed, instead of A in **4**, where must the force of 1 lb. be applied?

6. The point A being fixed, and forces of 3 lbs., 4 lbs., 5 lbs., 6 lbs., 7 lbs. acting along BC, CD, DB, through C parallel to DB, through the middle points of AB and DC respectively, find the point in AD, produced if necessary, nearest to A, at which a force of 2 lbs. will keep the cardboard at rest.

7. Forces of 1 lb. and 2 lbs. act along AB and DC respectively; what point of the cardboard in AD must be fixed that equilibrium may be maintained?

8. Forces of 3 lbs., 4 lbs., and 5 lbs. act along AB, BC, CD; find the points of the cardboard (i) in AD produced if necessary, (ii) in DC produced if necessary, which must be fixed in order to keep the cardboard from moving.

9. Forces of 4 lbs., 5 lbs., and 6 lbs. act along AB, BC, CA; find the points in which the line of action of their resultant cuts AD and DC.

10. Forces of 2 lbs., 2 lbs., and $2\sqrt{2}$ lbs. act along AB, BC and CA respectively; shew that it is impossible to keep the cardboard at rest by fixing one point only.

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11. An equilateral triangle ABC is drawn on a piece of cardboard placed on a smooth table, each side being 3 inches; forces of 3 lbs., 4 lbs., and 5 lbs. act along AB, BC and CA respectively; if the centre of gravity of the triangle be fixed, what is the least force which acting at A will keep the triangle at rest?

12. In Question **11** find a point in the perpendicular through A to BC which if fixed will keep the cardboard on which the three forces act at rest.

13. Forces of 3 lbs. each act along AC, CB, BA of the triangle ABC in question **11**; shew that the cardboard cannot be kept at rest by fixing one point only.

14. If in Question **11** the points A and B are fixed, what forces perpendicular to the line AB act on the constraints?

15. A rod AOB, such that AOB is a right angle, is fixed at O and is in a vertical plane; it is in equilibrium when weights P and Q are suspended from A and B, and OA is inclined at an angle of 60° to the horizon; find what change will require to be made in the force at B so that the rod may rest with OA inclined to the horizon at an angle of 30°.

16. A straight uniform rod ACB of weight W has the point C fixed, and weights P and Q are fastened at A and B respectively; shew that if the rod be at rest

$$AC : CB = 2Q + W : 2P + W.$$

17. A triangular board ABC weighing 3 lbs. with its plane vertical is hinged at A; what vertical force must act at the middle point of BC to keep it at rest?

18. The centre of gravity of an equilateral triangle is fixed and its plane is vertical; weights of 3 lbs., 4 lbs., 5 lbs. are fixed to the angular points; find the horizontal force which must act along the side joining the 4 lbs. and 5 lbs., that the 3 lbs. may rest vertically above the centre of gravity.

19. A ladder AB weighing 60 lbs. whose C. G. is 10 ft. from A has the point A fixed to the ground; what force must a man 6 ft. high be able to apply to the ladder to raise it to a vertical position supposing he applies the force at the point in the ladder which is 6 ft. from the ground?

20. If the sides of a triangle be taken two and two, to represent forces, acting in each case from the point of intersection of the sides, prove that there is one point about which each of the three pairs will balance and find the point.

CHAPTER XVII.

TENSIONS OF STRINGS AND OF RODS.

198. A string or a rod will bear a tension which is very great when compared with its own weight; hence the weight of a string in a machine is usually neglected.

Such strings are said to be **light**. It will be always understood that a string is *light* unless it is expressly stated to be otherwise.

199. The tension of a string exemplifies in a remarkable way the truth of Newton's third Law.

Take a piece of light string and pull at one end with each hand. Then the string applies a force, say for example of 2 lbs., to each hand. Fix your attention upon any point in the string. The string is said to be *tight* at that point; there is a *stress* at that point consisting of two equal and opposite forces each equal to 2 lbs. weight. This is the case at *every point* of the string. This *stress* is the *tension* of the string.

200. At the point at which a string is attached to a body the stress acts between the body and the string; one of the two equal and opposite forces of this stress urges the body towards the string; the other urges the string towards the body. We usually confine our attention to the first of these two forces.

201. Similar remarks hold good with regard both to the pull and the thrust of a light rod.

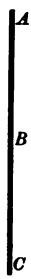
202. When a string is **heavy** the stress at each point is not the same for all points in its length.

Therefore when a string is heavy its tension is **not constant** along its whole length.

Example. One end of a uniform string 25 ft. long *which weighs* 4 lbs. per linear foot, *is attached to a hook; the string hanging vertically down supports a weight of* 500 lbs. *attached to its lower end; what are the tensions of the string at each end and at its middle point?*

Let ABC be the string, A its highest, B its middle and C its lowest point.

The tension at C must be the force necessary to keep the weight W 500 lbs. in equilibrium.



The weight is a rigid body acted on only by its own weight vertically downwards and by the pull of the string; which must therefore be a force of 500 lbs. vertically upwards.

To find the tension at the middle point of the string, consider the lower half of the string and the weight W as forming one rigid body, (the equilibrium would not be disturbed if they were actually to become rigid). Then as before the pull of the string upon its lower half must be 500 lbs. + the weight of $12\frac{1}{2}$ ft. of string; that is 550 lbs. This is the required tension.

To find the tension at the highest point, consider the whole string and the weight as a single rigid body and we find the tension to be 500 lbs. + the weight of the string; that is, 600 lbs.

203. It should be noticed, that when a system of particles or bodies is in equilibrium, we may choose any portion of the system for consideration; then the external forces acting on that portion must satisfy the conditions for the equilibrium of a single rigid body.

That this is so is perhaps made more evident by pointing out, that the equilibrium when it exists, would not be disturbed if the selected portion of the system were *actually to become rigid.*

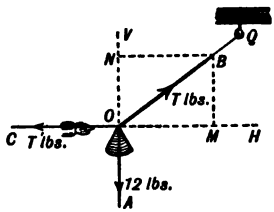
Example. A weight of 12 lbs. *is suspended from a fixed hook by a string; I tie a second string to the weight and by pulling horizontally cause the first string to be inclined to the vertical at an angle whose* cosine is $\frac{3}{5}$. *Find the* forces *applied by the strings.*

Let O be the point of the weight to which the strings are attached; let Q be the hook, P the hand holding the second string. Then we have three forces acting at O:

(i) the weight of 12 lbs. acting vertically downwards; represent this by OA:

(ii) the tension of the string along OQ; represent this by OB, and let it be T lbs.:

(iii) the tension of the string along OC which is horizontal; represent this by OC, and let it be T' lbs.



Then these three forces are in equilibrium.

Therefore the sum of their resolutes in any direction is zero.

Draw OH and OV horizontally and vertically.

Then cos $BOV=\frac{3}{5}$, and $\therefore$ cos $BOH=\frac{3}{5}$.

Take the sum of the horizontal resolutes of the forces.

The resolute of OC along OH is itself; viz. $-T'$ lbs.

The resolute of OA ..... is zero.

The resolute of OB ..... is $(T \cos BOH)$ lbs. $=\frac{3}{5}T$ lbs.;

therefore $-T'+\frac{3}{5}T=0$.

Take the sum of the vertical resolutes of the forces.

The resolute of OC along OV is zero.

The resolute of OA ..... is -12 lbs.

The resolute of OB ..... is $(T \cos BOV)$ lbs. $=\frac{4}{5}T$ lbs.;

$$\therefore \quad -12+\tfrac{4}{5}T=0;$$

$$\therefore \quad T=20,$$

and $$T'=\tfrac{3}{5}T=16;$$

$\therefore$ the tension of OQ is 20 lbs.: the tension of OP is 16 lbs.

[NOTE.—We here look upon a string as simply a mechanical means of applying force to the 10 lbs. weight. The force applied by a string to a mass is always **in the direction of the string**; also the magnitude of the force has nothing to do with the length of the string.]

EXAMPLES. XXXIX.

N.B.—The *length* of a string has of itself nothing whatever to do with its tension.

1. A weight of 140 lbs. hangs at the end of a rope 12 ft. long, whose other end is fixed; if the whole rope weighs 3 lbs., find the tension of the rope at a point 4 ft. from the weight.

2. A weight 112 lbs. hangs at the end of a rope 10 ft. long, whose other end is fixed; if the rope weighs 1 lb. per linear foot, find the tension of the rope at the point 4 ft. from the weight.

3. A chain weighing 3 lbs. per linear foot passes over a smooth small fixed pulley which is 20 feet from the ground; one end of the chain is coiled up on the ground vertically under the pulley, the other hangs vertically and is 10 ft. from the ground; what weight must be fastened to the chain to keep it at rest?

4. Shew that in Question 3 if the weight fastened to the end of the chain be too small it will run up to the pulley, if it be too large it will descend to the ground.

5. A chain weighing 1 lb. per foot passes over a smooth small fixed pulley† 50 feet from the ground, and one end is coiled up on the ground; the other end is held by a man at a point 30 ft. below the pulley; if the man pulls the chain over the pulley, what force must he exert? what is the pressure on the pulley?

6. Shew that in Question 5 when the end of the chain has left the ground the man's pull will gradually diminish; when will it cease altogether?

7. If a heavy body hang supported by three strings, two of which are vertical, prove that the third string must be vertical.

8. A heavy chain of length 12 ft. 8 in., and weighing 19 lbs., has a weight of 3 lbs. attached to one end, and hangs in equilibrium over a smooth peg; what length of it hangs on each side?

9. An endless chain 20 feet long, weighing 20 lbs., passes round a smooth fixed peg; two weights of 4 lbs. and 2 lbs. are fastened to points in the chain 5 ft. apart; what positions will the weights take up when hanging vertically under the peg in equilibrium, and what will be the tensions of the chain at the points 6 ft. from the peg?

10. An endless chain 16 ft. long, weighing 32 lbs., passes round a fixed smooth peg; three weights of 10 lbs. and 5 lbs. and 2 lbs. are fastened to the chain at points 2 ft. apart; where will the weights be when the chain hangs vertically from the peg in equilibrium? Find the tension of the chain at its lowest point, and at points 4 ft. from the peg.

† NOTE.—A smooth pulley or peg is here to be considered a mechanical means of altering the *direction* of the tension of a string without altering its magnitude.

11. A weight of 12 lbs. is fastened by a string to a hook; if I pull horizontally with another string fastened to the weight until the first string makes an angle whose cosine is $\frac{4}{5}$ with the vertical, with what force do I pull?

12. With a similar arrangement to that in Question **11**, if the weight is 16 lbs. and my horizontal pull is 12 lbs., find the tension of the other string.

13. If in a similar arrangement to that in Question **11** the inclination of the first string to the vertical is an angle whose cosine is $\frac{5}{13}$; when I pull horizontally with a force of 12 lbs., what is the weight?

14. A weight of 12 lbs. is supported by two strings each making an angle whose cosine is $\frac{4}{5}$ with the vertical; find the tensions of the strings.

15. A weight of 84 lbs. is supported by two strings, one making an angle whose cosine is $\frac{4}{5}$, and the other an angle whose cosine is $\frac{3}{5}$ with the vertical; find their tensions.

16. A weight of 204 lbs. is supported by two strings, making a right angle with each other, one of which makes an angle whose cosine is $\frac{5}{13}$ with the vertical; find their tensions.

17. A weight O of 84 lbs. is supported by two strings OA, OB fastened to two fixed hooks A and B such that $OA=4$ ft., $OB=3$ ft. and AB is horizontal and $=5$ ft.: find the tensions of the strings.

18. A weight O of 204 lbs. is supported by two strings OA, OB fastened to two fixed points A and B, so that $OA=5$ ft., $OB=12$ ft. and $AB=13$ ft. and is horizontal; find the tensions of the strings.

19. A weight O is suspended from a fixed hook Q by a string OQ; shew that if I apply a horizontal force to the weight O, I increase the tension of the string OQ.

20. Prove that if a uniform rod be suspended from a smooth peg by the loop of a string whose ends are fastened to the ends of the rod, the rod can only rest in a vertical or in a horizontal position.

SECTION V.

CHAPTER XVIII.

MACHINES.

204. We proceed to describe and as far as may be explain the principles of the following machines:

The Lever, the Pulley, the Wheel and Axle, the Inclined Plane, the Screw.

LEVERS.

205. DEF. A **lever** is a rigid rod moveable in one plane, about a fixed point in the rod.

206. The fixed point is called the **fulcrum.**

207. It is understood that two forces act on the lever besides the reaction of the fulcrum; these two forces are often called the **power** and the **weight.**

208. The **arms** of a lever are the *perpendicular distances* of the lines of action of the power and the weight from the fulcrum. [See Art. 120.]

209. The weight of the lever itself is generally inconsiderable compared with the forces acting upon it and accordingly the weight of the lever is often neglected.

It is then said to be *light.*

210. PROP. *To prove the principle of the Lever;* namely, that when a lever is in equilibrium

the Power × its arm is equal to the Weight × its arm.

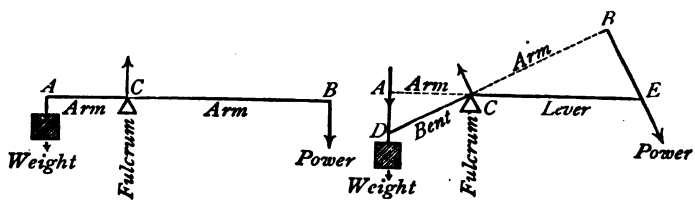
When the Power and the Weight are two parallel forces and the lever is straight the arms may be measured along the lever. Otherwise, the arm is the *perpendicular* distance of the line of action of the force from the fulcrum.

The point of the lever in contact with the fulcrum is *fixed;* therefore the fulcrum will apply to the lever whatever force is necessary to keep that point of the lever at rest.

The remaining forces will be in equilibrium with this constraining force provided the sum of the moments about that point is zero. [See Art. 196.]

That is, provided

the Power × its arm = the Weight × its arm.



211. It has been customary to divide levers into three classes.

In the **first class**, the fulcrum is between the Power and the Weight.

In the **second class**, the Weight is between the Power and the fulcrum.

In the **third class**, the Power is between the Weight and the fulcrum.

But these *classes* have no importance beyond the historical interest.

Examples of the first class: a crowbar; a poker raising the coals using the bar of a grate as a fulcrum; the handle of a pump. Double levers: scissors; pliers, nippers and pincers.

Examples of the second class: a wheelbarrow; an oar (in which the blade is the fulcrum and is supposed to be practically at rest). Double levers: nutcrackers.

Examples of the third class: the human arm, when put into a horizontal position with the palm of the hand upwards and supporting a weight, is a good example; the power is the biceps muscle which is attached to the arm near the elbow, and the elbow joint is the fulcrum. Double levers: a pair of fire tongs, as usually used; sugar tongs.

ON MECHANICAL ADVANTAGE.

212. One of the most important problems which an Engineer has to solve, is how to raise heavy weights.

With a lever, the ratio of the two forces, the Power and the Weight, may be made (by properly arranging the length of the arms) as small or as great as we please.

Hence we may with a lever *support* as great a weight as we please with as small a force as we please; provided the fulcrum is *fixed* and the lever is strong enough.

213. DEF. The capacity for counteracting a large force with a smaller one is called **mechanical advantage.**

It is customary to say that a machine possesses mechanical advantage when the *weight* supported is greater than the *power* by which it is supported.

The mechanical advantage of such a machine is said to be *measured* by the ratio of the weight to the power.

For example, a lever of the first class has mechanical advantage when the power-arm is longer than the weight-arm. A lever of the second class has always mechanical advantage. A lever of the third class is always at a **disadvantage** mechanically.

214. It must however be noticed that the advantage possessed by a machine is only *statical.* When a lever raises a weight it will be found that the *Work* done on the Weight is exactly equal to the work done by the Power.

For example, if a weight of 100 lbs. is supported by a force of 5 lbs. with a lever of the first class, the arm of the power is 20 times as long as the arm of the weight, and it therefore must move through 20 times the distance through which the weight moves.

What is gained in force is lost in distance moved.

EXAMPLES. XL.

1. A lever with the fulcrum between the Power and the Weight has its arms 5 ft. and 10 ft. respectively; if a power of 20 lbs. acts at the extremity of the longer arm, what weight can the lever support?

2. A lever with the fulcrum at one end has arms such that one is 3 ft. longer than the other; if the power is 10 times the weight, what is the length of the lever?

3. A lever with the power in the middle (i.e. *between* the fulcrum and the Weight); the power is $3\frac{1}{2}$ times the weight and the pressure on the fulcrum is 18 lbs.; what is the weight?

4. A heavy uniform rod 10 ft. long is used as a lever, and the fulcrum is 3 ft. from the end; the power is 1 lb. and the weight is 4 lbs.; what is the weight of the rod?

5. A wheel-barrow is 5 ft. long and exclusive of the wheel it weighs 20 lbs., the centre of gravity being 2 ft. from the axle of the wheel; how near the axle must a weight of 30 lbs. be placed that a man may gain mechanical advantage by using the wheel-barrow?

6. A cube weighing 1 ton stands on a horizontal plane. A man of 12 stone wishes with a crow-bar 5 ft. long to raise the middle of one edge of the cube from the ground; how near the end of his crow-bar must he put his fulcrum?

7. A pair of nutcrackers is $4\frac{1}{2}$ inches long, and a nut is placed $\frac{5}{8}$ of an inch from the hinge; what pressure applied at the ends will crack the nut if a weight of $20\frac{1}{4}$ lbs. when simply placed on the top of the nut will crack it?

8. A heavy uniform beam 7 ft. long rests on two supports, one at one end and the other $5\frac{1}{2}$ feet from that end; the greatest weight that can be hung on at the other end without disturbing the equilibrium is 16 lbs.; find the weight of the beam.

9. Explain two different ways of arranging a rod 15 ft. long as a lever which will lift a weight of 24 lbs. with a force of 12 lbs.

10. A straight rod AF without weight is divided in points B, C, D, E so that $AB : BC : CD : DE : EF$ as $1 : 3 : 5 : 7 : 9$, and weights of 1, 2, 3, 4, 5 lbs. are placed at the points B, C, D, E, F respectively; shew that if G be the point in the rod at which a fulcrum would support it, $AG : GF = 3 : 2$.

11. Find the length of a lever of the second kind that a power of 5 lbs. may support a weight of 12 lbs., and that their points of application may be 1 ft. apart.

12. The shorter arm of a lever is 7 inches long; the lever is in equilibrium when weights of 5 lbs. and 8 lbs. are suspended from its arms; find the length of the other arm.

13. The arms of a light lever of the first kind are unequal, and a body when suspended from one end is balanced by a weight of 18 lbs. at the other; when suspended from the longer end it is balanced by a weight of 20 lbs.; what is the ratio of the arms of the lever, and what is the weight of the body?

14. A straight lever whose length is 5 ft. and weight 10 lbs. has its fulcrum at one end; weights of 3 lbs. and 6 lbs. are fastened to it at distances 1 ft. and 3 ft. from the fulcrum, and it is kept horizontal by a vertical force at the other end; find the pressure on the fulcrum.

15. If in Question **14** the force keeping the lever horizontal were inclined at an angle of 30° to the horizon, what would the force be, and what would be the pressure on the fulcrum?

16. AOB is a bent lever whose fulcrum is O, and whose arms OA, OB are equal and straight; it is in equilibrium with OA horizontal when weights P and Q are suspended from A and B; find the change which must be made in the weight suspended at B that the lever may be in equilibrium when P is suspended at A and OB is horizontal.

17. A lever has its fulcrum in the middle (i.e. between the Power and the Weight), and a weight W fastened to one end is supported by a force of P lbs. at the other; if the ends are interchanged the necessary force to balance W is a force of Q lbs.; prove that $W=\sqrt{(PQ)}$.

18. The arms of a bent lever are 2 ft. and 3 ft. respectively; what force acting at an angle of 30° to the longer arm will balance a force of 30 lbs. acting at right angles to the shorter arm?

19. A bent lever whose arms are inclined at right angles to each other, and are 3 ft. and 4 ft. long respectively, is at rest under the action of forces of 16 lbs. and 12 lbs. acting at the extremities of the shorter and longer arms respectively; if the force of 16 lbs. acts at right angles to its arm, at what angle must the other force act and what is the pressure on the fulcrum?

20. A bent lever consists of two heavy uniform straight arms, whose lengths are 3 ft. and 5 ft. respectively; if the beam weighs 10 lbs. per foot, what weight must be suspended from the extremity of the shorter arm that the lever may balance with its arms equally inclined to the horizon?

21. AB and DEC are two light horizontal levers arranged so that B is vertically above C, and connected with it by a light inextensible string. The fulcrum of DC is at D; the weight W is placed at E so that DE is $\frac{1}{8}^{\text{th}}$ of EC; the 'weight' of AB is the tension of the string; the fulcrum of AB is at H, so that BH is $\frac{1}{10}^{\text{th}}$ of AH; what power acting vertically downwards at A will support 100 lbs. placed at E?

NOTE. The above **compound lever** illustrates the principle of the weighing-machine.

THE COMMON BALANCE.

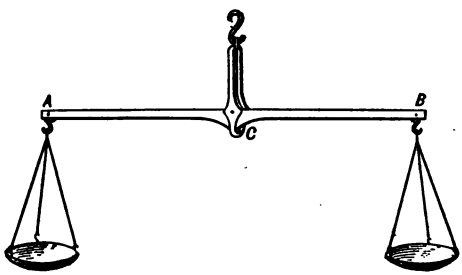
215. The common balance is a machine for testing whether two bodies have equal weights.

It is practically a lever with the fulcrum exactly in the middle of the beam.

216. The '**weights**' of a balance are bodies whose weights are some multiple or submultiple of the standard weight.

The weights are bodies weighing respectively 112 lbs., 56 lbs., 28 lbs., 14 lbs., 7 lbs., 4 lbs., 2 lbs., 1 lb., ½ lb., ¼ lb., 1 oz., ½ oz., ¼ oz., etc.

217. The following is a description of the balance and the method of using it.



The figure gives a simple form of the balance. The *beam AB* is supported by a fulcrum at *C*. The fulcrum is a short bar of steel or other hard substance whose section is wedge-shaped; this is fixed to the beam with its sharp edge downwards and passes through round holes in the support from which the beam is suspended; thus the beam is supported by the upward pressure of the support on its sharp edge called the *knife-edge*. Knife-edges are fixed to the beam at *A* and *B* and from them are suspended the scale-pans; so that the upward pressure of each of these knife-edges supports the weight of a scale-pan and of anything placed on it. The short bars forming the knife-edges are at right angles to the plane of the picture.

The body to be weighed is placed in one scale-pan, say *A*, which at first rests on a table, and weights are placed in the other scale-pan *B* also on the table; the beam is then raised by the hook until it is just horizontal, and both scale-pans just touch the table; it is then quickly raised a little further, so as to lift both scale-pans off the table. Then,

if the scale-pan *A* falls, the weights are too small; if the scale-pan *B* falls, the weights are too great; the trials are continued until the beam *hesitates* to turn one way or the other.

218. The **requisites** of a good balance are

(I) it must be **true**; the arms must be *exactly* equal, and the scale-pans of equal weight;

(II) it must be **sensitive**†; that is, it must *quickly* shew when the sum of the weights, which are placed in the scale-pans opposite to that in which is placed the body to be weighed, is *not* equal to the weight of the body;

The discussion of the means by which a balance may be made sensitive is really a *dynamical* question.

It will be found that in order that a balance may be sensitive it is necessary

(i) The knife-edges of the beam and of the pans must be very hard and must rest on a hard surface.

(ii) The arms must be as long as possible.

(iii) The beam must be as light as possible.

(iv) The C.G. of the beam must be near the knife-edge on which it rests.

(III) it must be **stable**; that is, it must not move very far from its mean position, when the weight of the body and of the weights differ.

This is also a dynamical question; but we may point out here that, when the knife-edge on which the beam rests is above the straight line joining the knife-edges of the scale-pans (as it usually is), then the nearer the knife-edge of the fulcrum is to the straight line joining the knife-edges of the scale-pans, the more sensitive and the less stable will the balance be.

The further *above* this line the fulcrum is, the more stable it will be. Hence for purposes requiring great accuracy the three knife-edges are made nearly or exactly in a straight line; while for the rougher purposes of weighing heavy goods the knife-edge of the fulcrum is placed a sensible distance above the line joining the knife-edges supporting the scale-pans.

† The word *sensible* is often used in this sense.

219. It is not sufficient for the truth of a balance that the beam should remain horizontal when raised with the scale-pans empty, for the scale-pans might in that case be of unequal weight and the arms also unequal.

220. To test the truth of a balance, we first see that the beam balances when the pans are empty; then taking two equal weights we put one in each of the scale-pans; if the beam again balances, then the balance is true.

For, let a, b be the lengths of the arms; W, W' the weights of the scale-pans; ω the weight of the equal weights; then by the first experiment $W \times a = W' \times b$,

by the second $(W+\omega) \times a = (W'+\omega) \times b$;

$\therefore\ \omega \times a = \omega \times b$;

$\therefore\ a = b$, and $\therefore$ also $W = W'$.

Or, we may (having first tested when the pans are empty) find the weight of a body Q when placed in one scale-pan to be W_1, and when placed in the other to be also W_1; then the balance is true.

EXAMPLES. XLI.

1. In a balance which rests in a horizontal position when unloaded, but whose arms are of different lengths, a body when weighed in one scale appears to weigh W lb. and when weighed in the other W' lb.; prove that its true weight is $\sqrt{(WW')}$.

2. A grocer has to weigh out to a customer a certain weight of tea; he knows that his balance is in the condition described in Question **1**, so he weighs one half of the quantity of tea required, and then puts his weights into the other scale and weighs the other half in the other pan; has he got the proper weight of tea, or is he a gainer or loser?

3. In a balance such as that described in Question **1**, the apparent weights of a body are $42\frac{1}{4}$ lbs. and 49 lbs., and the whole length of the beam is $2\frac{1}{4}$ ft.; find the length of each arm.

4. In a false balance, the arms being of unequal length, a weight is measured in one scale by P lbs. and in the other by Q lbs.; shew that the arms are to one another as $\sqrt{P} : \sqrt{Q}$.

5. If the arms of a false balance are 2 ft. and 2 ft. 1 in. respectively, what is the true weight of a body which appears to weigh 10 lbs. when placed in the scale at the end of the shorter arm?

6. When one of the scales of a common balance (which is otherwise true) is loaded, a body appears to weigh W lbs. when placed in one scale and W' when placed in the other scale; prove that its true weight is the Arithmetic mean between W and W'.

7. If the apparent weights in Question **6** are 18 oz. and 20 oz., find the weight with which one of the scales is loaded.

8. Shew that a balance is (other things being equal) more sensitive as the arms are longer.

9. If the beam of a false balance is uniform and heavy, shew that the arms are proportional to the differences between the true weight and the apparent weights of a body.

10. The arms of a false balance are without weight, and one arm is longer than the other by one-ninth part of the shorter arm; and in using it the substance weighed is put as often into one scale as in the other, shew that the seller loses five-ninths per cent. on his transactions.

11. A tradesman's balance has arms whose lengths are 11 in. and 12 in. respectively, and it rests horizontally when the scales are empty; if he sells two separate pounds of tea each at 2*s.* 9*d.* per lb., putting his weights into different scales for each transaction, find whether he gains or loses, and how much.

12. One pound is weighed at each end of a false balance and the sum of the apparent weights is 2 lb. 2 oz.; what is the ratio of the lengths of the arms?

13. A balance has its arms unequal in length and in weight. A certain article appears to weigh Q_1 lbs. when placed in one scale-pan, and Q_2 lbs. when placed in the other; a certain other article appears to weigh R_1 lbs., R_2 lbs. respectively; shew that the weight of the article which appears to weigh the same in whichever scale it is put is

$$\frac{Q_1R_2 - Q_2R_1}{Q_1 - Q_2 - R_1 + R_2} \text{ lbs.}$$

221. **The common steelyard.** The common steelyard is an instrument for weighing goods.

It consists of a straight steel lever AB having a fulcrum at a fixed point C near one end A.

A weight of P lbs. is arranged as a ring D which can slide along the arm CB. At A is a hook or scale-pan on which can be placed the article whose weight is to be ascertained. The arm CB has certain marks and numbers engraved on it.

The weight of an article is obtained as follows:

The fulcrum is firmly supported and the steelyard is held in a horizontal position while the article is placed in the scale-pan. Then the ring D is shifted until by repeated trials the position is found for it at which, when the whole is left to the support of the fulcrum alone, the steelyard *hesitates which way to turn*. The mark at D engraved on the rod OB indicates the weight of the article.

The **graduation** of the steelyard is the process by which the maker ascertains where to put the proper marks on the rod CB.

222. PROP. *To graduate the common steelyard.*

Let the weight of the rod and scale-pan together be M lbs.; and let their joint centre of gravity be at G.

[The weight of the scale-pan acts upon the rod exactly as the weight of an equal body *fixed to the rod* at A.]

We first obtain the zero point O by trial.

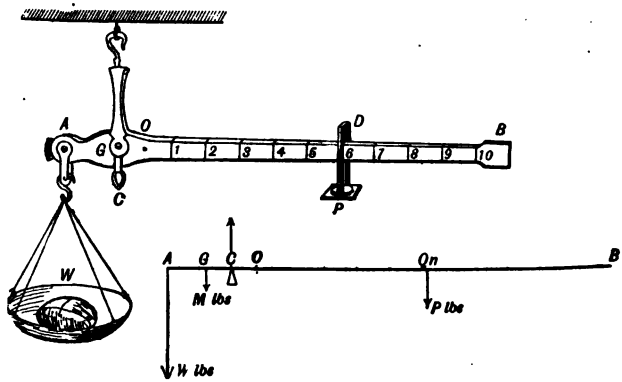
That is, the ring D, which weighs P lbs., is shifted until a point O is found, such that the steelyard balances with the ring at O, when no article at all is in the scale-pan.

Then $$M \times GC = P \times OC' \quad \text{.....(i).}$$

Now suppose a weight of W lbs. to be placed in the scale-pan.

Let the ring D be shifted to a point Q_n at which it is found that the steelyard balances.

Then we have a force of W lbs. acting vertically downwards on the steelyard at A, a force of M lbs. acting verti-



cally downwards at G and a force of P lbs. acting vertically downwards at Q_n.

These forces P, M, W and a vertical force upwards at C are in equilibrium.

Taking the moments about C of all these forces we have

$$W \times AC + M \times GC - P \times CQ_n = 0 \ldots\ldots\ldots (ii).$$

But by our first experiment we have

$$M \times GC - P \times OC = 0 \ldots\ldots\ldots\ldots (iii);$$

$\therefore$ putting $P \times OC$ for $M \times GC$ in (i), we have

$$W \times AC + P \times CO - P \times CQ_n = 0,$$

that is
$$P(CQ_n - CO) = W \times AC,$$

or
$$P \times OQ_n = W \times AC,$$

$$\therefore OQ_n = \frac{W \times AC}{P}.$$

From this, since AC is known, and O has been found, the different positions of Q_n for different values of W can be found.

223. Suppose now that $W = 1$ lb.; then $OQ_1 = \frac{AC}{P}$;

$\therefore$ to find Q_n for n lbs., we have $OQ_n = n \times OQ_1$.

That is, in order to graduate a steelyard to weigh lbs., we must find by trial the point O of zero graduation, and also by a second trial the point Q_1 for 1 lb., and mark their positions on the rod. Next we find by measurement points Q_2, Q_3, Q_4, etc. such that $OQ_1 = Q_1Q_2 = Q_2Q_3$, and so on, and engrave on the rod the numbers 2, 3, 4, etc. at those points.

Then, when an article weighing a number of lbs. say 4, is placed in the scale-pan, the steelyard balances when the ring is at the point on the rod which is marked 4.

NOTE. The student should observe that if the lengths OQ_1, Q_1Q_2,... be each subdivided into any number of equal parts, say 16, then the steelyard is graduated for sixteenths of a lb.

Example. In a certain Common Steelyard the weight of the rod and scale-pan is 4 lbs.; *the length of* AC *is* 2 inches; G *lies between* A *and* C, *and* GC *is* $\frac{1}{2}$ in.; $P = 2$ lbs.; *graduate the steelyard.*

We first find O.

We have $M = 4$ lbs.; $GC = \frac{1}{2}$ in.; $P = 2$ lbs.;

$\therefore$ by (i) $4 \times \frac{1}{2} = 2 \times OC$,

$\therefore OC = 1$ inch.

The point O must be marked on the bar at a distance 1 inch from C.

Next we find OQ_n when $n = 1$ lb.

We have $$OQ_n = \frac{n \times AC}{P} = \frac{1 \times 2}{2} = 1.$$

Thus the distance between each graduation for lbs. is 1 inch.

Hence marks are engraved on the bar at equal distances of an inch from the point O, and the numbers 1, 2, 3, etc. placed near them. These spaces are then subdivided into sixteenths and the machine is then graduated to weigh lbs. and oz.

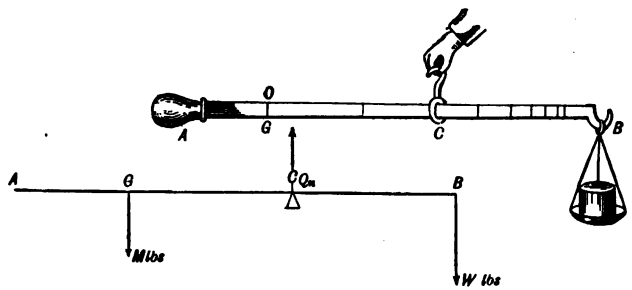
224. **The Danish steelyard.** The Danish steelyard consists of a lever AB whose fulcrum C is moveable.

At one end A is a lump of metal, and at the other B, a hook or scale-pan.

The weight of an article placed in the scale-pan is ascertained by observing the point at which the fulcrum must be placed in order that the whole should balance when supported by the fulcrum.

225. PROP. *To graduate the Danish steelyard.*

The position of O, the zero point of graduation, is the centre of gravity G of the rod and scale-pan. [For when the ring is at G the steelyard balances when there is no weight in the scale-pan.] This must be ascertained by trial and carefully marked on the rod.



Let M lbs. be the weight of the rod and scale-pan.

Let W lbs. be placed in the scale-pan, and let Q_n be the corresponding position of the fulcrum.

Then the three forces M, W and a vertical force upwards at C are in equilibrium.

Hence, taking the moments of all these forces about C, we have $W \times Q_nB - M \times GQ_n = 0$.

From this result, since M is known, the position of Q_n for different values of W can be determined.

It will be seen that Q_n divides GB into parts which are in the ratio of W to M.

Example. A Danish steelyard and its pan weighs 4 lbs., shew how to graduate it so as to weigh lbs.

We have to find the points on the rod at which we must put the marks 1, 2, 3, etc. which shall indicate that when the steelyard balances with the fulcrum say at 2 the weight in the scale-pan is 2 lbs.

We have $W \times Q_nB - M \times GQ_n = 0$,

or $$GQ_n = \frac{W}{M} \times Q_nB.$$

Also $M = 4$ lbs.

Hence $GQ_n = \frac{1}{4} W \times Q_n B$.

First, let $W = 1$ lb.; then $GQ_n = \frac{1}{4} Q_n B = \frac{1}{5}$ of GB;

hence, if we divide GB into *five* equal parts, GQ_n contains one of those parts.

Next, let $W = 2$ lbs.; then $GQ_n = \frac{2}{4} Q_n B = \frac{2}{6}$ of GB;

hence, if we divide GB into *six* equal parts, GQ_n contains two of those parts.

Next, let $W = 3$ lbs.; then $GQ_n = \frac{3}{4} Q_n B = \frac{3}{7}$ of GB;

hence, if we divide GB into *seven* equal parts, GQ_n contains three of those parts. And so on;

When $W = 4$ lbs., $GQ_n = \frac{4}{8}$ of GB;

when $W = 5$ lbs., $GQ_n = \frac{5}{9}$ of GB; etc.

226. NOTE. In the Balance and the Steelyards the scale-pans are treated as forming part of the beam; the student should notice that it would make no difference to the argument if the scale-pan were replaced by an equal weight *fixed to the beam at the point of suspension.*

The student will notice that in the Balance and Steelyards it is important that the *points of suspension* of the scale-pans should be *fixed in the beam.* Hence in the steelyards the scale-pan should be suspended from a knife-edge *fixed to the beam* as in the balance.

EXAMPLES. XLII.

1. A common steelyard is formed of a uniform rod 1 foot long, the fulcrum being 1 inch from one end; the sliding weight and the weight of the rod are each 1 lb.; find the least weight that can be weighed with it, and the distance between the graduations for pounds.

2. If in the steelyard of Question **1** the sliding weight be changed into one of 2 lbs., find what error will be made by using the old graduations as 2 lb. graduations.

3. If the distance of the C. G. of the beam of a common steelyard from the fulcrum is 2 inches, the moveable weight 4 oz., and the weight of the beam 2 lbs., find the distance of the zero of graduations from the centre of gravity.

4. A uniform rod 2 feet long, weighing 3 lbs., is to be used as a steelyard; the fulcrum is 2 inches from one end of the rod and the sliding weight is 1 lb.; find the greatest and least weight that can be determined by the machine, and the distance between the 1 lb. graduations.

5. The errors of a certain false steelyard are these: the distance of the zero point from the fulcrum is too great by a distance a, and every one of the distances between consecutive graduations is too long by a distance b; shew that the only weight which is correctly indicated by the instrument is $\frac{a}{b}P$ where P is the moveable weight.

6. A common steelyard is 3 ft. long, the C. G. of the steelyard is 5 inches from one end, and the fulcrum is 6 inches from the same end; the weight of the steelyard is 8 lbs., and the moveable weight is $1\frac{1}{2}$ lbs.; find the position of the zero of graduation and the distance between the lb. graduations.

7. If the moveable weight for which a common steelyard is constructed is 1 lb., and a tradesman substitutes a weight of 2 lbs., using the same points of graduation as before, but doubling the value indicated by the marks, shew that he defrauds his customers if the C. G. of the steelyard is in the longer arm, and himself if it is in the shorter arm.

8. Where must be the C. G. of a common steelyard that any moveable weight may be used with it, the marks of graduation indicating multiples of the moveable weight?

9. In a Danish steelyard the zero of the graduations is 12 inches from the end at which the body to be weighed is attached, and the weight of the beam is 8 oz.; find the position of the graduation corresponding to a weight of 16 oz.

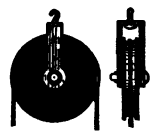
10. If the bar of a Danish steelyard balance when the fulcrum is halfway between the first and second graduations, shew that the weight then in the scale is $\frac{1}{8}$ of the weight of the bar.

11. A weight of 4 oz. is in equilibrium on a Danish steelyard when the fulcrum is 6 inches from the end to which the weight is attached; a weight of 8 oz. is in equilibrium when the fulcrum is 4 inches from the end; find the C. G. and the weight of the instrument.

12. The weight of a Danish steelyard is 1 lb. and the nearest distance of the fulcrum from the end from which the weight is to be suspended is 1 inch, and the distance between the above position of the fulcrum and the C. G. is 3 ft.; what is the greatest weight that can be weighed with the machine?

PULLEYS.

227. A pulley is a machine for changing the direction of the pull of a string without sensibly changing its magnitude.



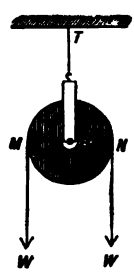
It consists of a small wheel of metal or wood whose axle is held by a frame called a **block.**

The word *pulley* is often used to indicate *the pulley and the block;* so that when we speak of a *fixed pulley*, we shall mean a pulley in a fixed block.

A moveable pulley generally indicates that the block of the pulley is not fixed, but has a string fastened to it; the pull of this string is in equilibrium with the pulls of the string which passes through the pulley.

It is usual to consider that the weight of the string is so small, compared with the other weights considered, that it may be omitted in our calculations.

Example i. *To find the conditions of equilibrium of a single fixed pulley, the strings being parallel.*



First, let the weight of the pulley be neglected.

Let the string passing round the pulley support a weight W at each extremity. These weights must be equal, because the pulley is smooth.

The block is fastened to a support by a string; let the tension of this string be T.

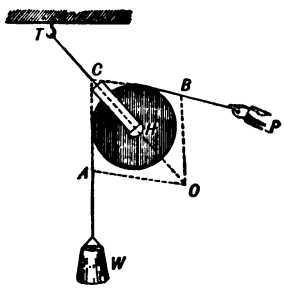
The pulley is now acted on by three parallel forces W, W, T which are in equilibrium.

Therefore $T = W + W = 2W$.

Next, consider the weight of the pulley to be w.

Then there are four parallel forces, three of which are like, viz. W, W and w; $\therefore\ T = 2W + w$.

Example ii. *The block* H *of a pulley is fastened to a fixed beam* T *by a string; the end of another string, which goes over the pulley, is fastened to a weight* W; *the other end I hold in my hand, as in the figure; with what force do I pull? and what is the tension of the strings?*



The tension of the string which is fastened to the weight, is equal to W, the weight itself.

This tension is unaltered after passing round the pulley; therefore the force with which I pull is equal to W.

First, suppose the weight of the pulley to be neglected.

Then the only forces acting on the pulley are the tensions of the strings.

These forces are in the direction of the strings.

Therefore if a parallelogram $ACBO$ be drawn as in the figure, AC represents W; BC represents W, and CO represents the tension of the string TC.

Then if the angle $BCA = 2\theta$, we have $CO = 2AC\cos\theta$.

Therefore the tension of CT is $2W\cos\theta$.

Next, suppose the weight of the pulley taken into consideration.

In this case there are *four* forces acting on the pulley: (i) its weight, (ii) and (iii) the two equal tensions of the strings PB, WA; (iv) the tension of the string TC. If the first three forces are given we can by Art. 91 find the magnitude and direction of the tension of the string TC.

228. In the simpler problems on pulleys, the blocks are usually so arranged that the strings are all vertical; in which case the problem is one on parallel forces only.

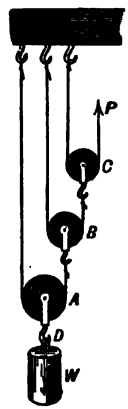
There are three arrangements of pulleys usually explained in elementary text-books on Statics. They consist of separate blocks so arranged that the tension of one string caused by a force called the **Power**, supports a larger vertical force called the **Weight**.

We shall consider the strings to be all vertical.

229. PROP. *To find the relation between the* Power *and the* Weight *in a system of pulleys, in which each pulley hangs in the loop of a separate string, one end of which is fastened to a fixed beam; all the strings being parallel.*

[This system is called THE FIRST SYSTEM OF PULLEYS.]

The system is shewn in the figure.



First, neglect the weight of each pulley.

Let P be the Power, and W the Weight.

Let A, B, C... be the pulleys.

Let T_1, T_2, T_3... be the tensions of the strings which go round the pulleys A, B, C... respectively.

Then, the forces on the pulley A are three parallel forces, W downwards and T_1 and T_1 upwards;

$$\therefore\ 2T_1 = W.$$

The forces acting on the pulley B are three parallel forces, T_1 downwards and T_2 and T_2 upwards;

$$\therefore\ 2T_2 = T_1.$$

Similarly, $$2T_3 = T_2,$$

and so on.

$$\therefore\ W = 2T_1 = 4T_2 = 8T_3 = \ldots = 2^n T_n.$$

And when there are n pulleys $P = T_n$;

therefore $$W = 2^n T_n = 2^n P;$$

or $$P = \frac{W}{2^n}.$$

Next, let the weights of the pulleys A, B, C... be w_1, w_2, w_3 ...

Then, if there are n pulleys, in order to support the pulley A, the power must be increased by $\frac{w_1}{2^n}$;

to support the pulley B, the power must be increased by

$$\frac{w_2}{2^{n-1}};$$

and so on.

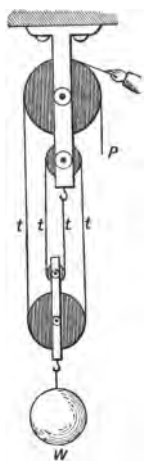
Hence, the power P^1 necessary to support both W and the n pulleys is given by

$$P^1 = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \ldots + \frac{w_{n-1}}{2^2} + \frac{w_n}{2}.$$

230. PROP. *To find the relation between the* Power *and the* Weight *in the system of pulleys in which all the pulleys are arranged in two blocks, one block fixed, the other moveable; the same string going round all the pulleys, and all the strings being parallel.*

[This system is called THE SECOND SYSTEM OF PULLEYS.]

The figure gives two pulleys to each block.



Since the *same* string goes round each pulley, the tension of every portion of the string is the same. Let t be the tension of the string. Then $t = P$ (the Power). Let W be the Weight, and let w be the weight of the lower block. The equilibrium would not be changed if the lower block, the weight, and the strings touching it were all to be considered as one rigid body. This rigid body is acted on by four equal parallel forces, each equal to t acting vertically upwards, and by its weight $W+w$ acting vertically downwards;

$$\therefore\ W + w = 4t = 4P.$$

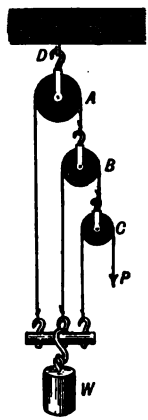
Similarly, when there are a number of pulleys so arranged that n strings pass upwards from the lower block, we have

$$W + w = nP.$$

NOTE. In practice the blocks are usually made so that the pulleys are all side by side. In this case the strings cannot be all exactly parallel. When the distance between the blocks is not very small, the angle between any two strings will be very small, and the results obtained from considering the strings parallel will be sensibly correct.

231. PROP. *To find the relation between the* Power *and the* Weight *in a system of pulleys in which each pulley is supported by a separate string, which, after passing over the pulley above it, has its other end fastened to the weight; all the strings being parallel.*

[This system is called THE THIRD SYSTEM OF PULLEYS.]



The figure gives a system of three pulleys.

Let P be the Power and W the Weight.

First, neglect the weights of the pulleys.

Let T_1, T_2, T_3,... be the tensions of the strings passing round A, B, C... respectively.

Let T be the strain on the hook at D from which the pulley A is suspended.

Then, considering the whole system as a single rigid body, the forces acting upon it are the pull T vertically upwards and W and P vertically downwards;

$$\therefore\ T = W + P.$$

But $T = 2T_1$, $T_1 = 2T_2$, $T_2 = 2T_3$, and so on, as in the First System.

$\therefore$ when there are n pulleys,

$$T = 2^n T_n = 2^n P;$$

$$\therefore\ 2^n P = W + P;$$

$$2^n P - P = W,$$

or
$$(2^n - 1)\, P = W;$$

$$\therefore\ P = \frac{W}{2^n - 1}.$$

Next, let the weights of the pulleys B, C... be w_2, w_3, ...

Then the *Weight* which can be supported by the weight of w_2 is $(2-1)\, w_2$;

the *Weight* which can be supported by the weight of w_3 is $(2^2-1)\, w_3$; and so on.

For W_3 takes the position of P in a system of two pulleys.

Hence we have the relation between P and W' (the Weight which can be supported by weights of the pulleys and P together)

$$W' = (2-1)w_2 + (2^2-1)w_3 + \ldots + (2^{n-1}-1)w_n + (2^n-1)P.$$

232. The student should notice that the First and Third Systems of Pulleys are practically the same; the one is simply the other inverted.

If page 176 or page 178 be inverted, this statement will be evident.

233. This System (I. and III.) is not much used for the raising of weights; it is too complicated; it is however often used on board ships where it is necessary to maintain a considerable tension on a certain rope.

The Second System of Pulleys is much used for the raising of weights in conjunction with the wheel and axle. The spare rope as it is pulled in from the pulleys is conveniently coiled on the wheel.

234. The **mechanical advantage** of a system of pulleys is measured by the ratio of the Weight to the Power.

It must not be supposed that any more *work* is done on the Weight than is done by the Power.

In the first system, in which (neglecting the weights of the pulleys) $W = 2^n P$, it will be found that for W to ascend 1 foot, P must descend 2^n feet.

Therefore $W \times$ (distance ascended by W) $= P \times$ (distance descended by P).

This is an instance of the principle that no machine can produce more work than is put into it from outside.

235. We have assumed that when a string passes round a pulley, the tensions of the parts of the string on opposite sides of the pulley are equal. This will be strictly true when the pulley is at rest, provided there is *no friction* at the axle of the pulley and the pulley is perfectly *circular*.

For, consider the forces acting on the pulley (not the block) in the figure; the forces all pass through the centre of the wheel except the two tensions.

The tensions may be supposed to be applied to the wheel at M and N, for equilibrium would not be disturbed if the string were to become fastened to the wheel at M and N.

Taking moments about the centre of the wheel we have by Art. 210, one tension × radius of wheel = the other tension × radius of the wheel; therefore the tensions are equal provided the radii of the wheel are all equal.

EXAMPLES. XLIII.

1. In a system of 4 pulleys of Art. 165, a weight of 16 lbs. is suspended from the lowest pulley, what is the power, the weight of the pulleys being neglected?

2. In a system of pulleys in which each pulley hangs by a separate string, the power is 10 lbs.; what is the weight if there are 3 pulleys whose weight may be neglected?

3. In the First System of Pulleys, $P=28$ lbs. and $W=16$ cwt.; find n.

4. In a system of three pulleys in which the string which passes round any pulley has one extremity fixed and the other fastened to the pulley next above it, a weight of 12 lbs. is suspended from the lowest pulley, which weighs 4 lbs.; the next pulley weighs 2 lbs., and highest 1 lb.; what weight will maintain equilibrium?

5. In a system of pulleys in which each pulley hangs from a fixed point by a separate string there are two pulleys, the upper weighing 1 lb., the lower 2 lbs.; what power would support a weight of 3 cwt.?

6. In the system of pulleys in which each pulley hangs from a fixed support by a separate string, the weights of three moveable pulleys, beginning with the lowest, are 4 lbs., 5 lbs. and 6 lbs. respectively; what weight will a power of 1 cwt. support?

7. How many pulleys each weighing 1 lb. must be used in a system in which each pulley has a separate string, one end of which is fastened to a fixed support, that a power of 2 lbs. may support a weight of 65 lbs. If the weight be allowed to descend 1 inch, how much will the power ascend?

8. In the system of pulleys of Art. 229, if there are three moveable pulleys, the lowest of which weighs 5 lbs., the middle one 4 lbs. and the highest 3 lbs., and the weight is 7 lbs., find the power.

9. If there be 3 pulleys in a system of Art. 229, and the pulleys each weigh W, find the power necessary to support the system when no weight is attached to the lowest pulley; what weight can be suspended from the lowest pulley when the power just found is doubled?

10. Find the weight of each of 4 equal pulleys in the system of Art. 229 that P may be equal to W when $W = 10$ lbs.: in this system what additional power must be applied in order to support a weight of 1 cwt. instead of 10 lbs.?

11. If the strings in a system of 3 weightless pulleys of Art. 229 are attached to a beam at points 1 foot apart, find the centre of the parallel forces applied by the strings to the beam.

12. A man standing on the floor pulls at the power end of a system of 3 pulleys of the First System; if the weight be four times the weight of the man, what is the pressure of his feet on the floor (neglecting the weights of the pulleys)?

13. If there are three pulleys (in the First System) of equal weight, find the weight of each pulley in order that a weight of 56 lbs. attached to the lowest pulley may be supported by a power of 7 lbs. 14 oz.?

14. If in the First System there be three pulleys each of weight w and $W = 2P$; find w.

15. In the system of pulleys in which there is only one string and there are 9 pulleys, the lower block containing five pulleys and weighing 10 lbs., what force will support a weight of 1 cwt.?

16. The cable by which Great Paul, the bell weighing 18 tons, was lifted to its place in the Cathedral tower, passed four times through each of two blocks of pulleys; find the lowest possible breaking strain of the cable.

17. In the system of pulleys in which one string only is used, the string is fastened to the lower block, which weighs 6 lbs. and contains 3 pulleys, what weight will a power of 10 lbs. support?

18. If in the system of the last Question the weight is 1 ton and the end of the string to which the power is usually applied is fastened to the lower block, what will be the tension?

19. What weight can be supported by a force of 2 lbs. by means of a system of 8 pulleys, 4 in one fixed block and 4 in a moveable block, the string passing over all of them when the lower block weighs 1 lb.?

20. In the system of pulleys in which the same string passes round all the pulleys the weight of the moveable block is 5 lbs., and the fixed and moveable block each contain two pulleys; find what weight a power of 1 cwt. will support.

21. Shew that in the system of pulleys in which there is only one string the tension is least when all the strings are parallel; shew also that all the strings cannot be parallel except one only.

22. How many times his own weight can a man raise with two blocks, one containing 4 pulleys and the other 5, each block weighing $\frac{1}{10}$th of the man's weight?

23. A man raises a weight of 1 ton by means of two blocks each containing three pulleys each weighing 10 lbs.; find the pull on the beam from which the upper block is suspended, and the least weight of the man.

24. Shew that in the Second System of Pulleys, unless the ratio of the weight of the lower block to the suspended weight be less than the number of strings in the lower block diminished by unity, there is no mechanical advantage.

THIRD SYSTEM.

25. In the system of pulleys in which each pulley has a separate string, one end of which is attached to the weight, if the power be 8 lbs. and the weight of the pulleys neglected, what is the weight when there are three pulleys?

26. If in the Third System there be three pulleys such that the weight of each pulley is equal to the power, shew that the power will support a weight 11 times as great as itself.

27. Shew that in a Third System of two pulleys, if each pulley weighs 1 lb., a power of 3 lbs. supports a weight of 10 lbs.

28. In a Third System of pulleys, if the weights of the pulleys are 1 lb., 2 lbs., 3 lbs. respectively, find the greatest weight and the least weight which can be kept in equilibrium by a power of 7 lbs., it being understood that the pulleys may be arranged in any order.

29. In the Third System if the weight be 2 cwt. 1 lb., and the power be 15 lbs. (neglecting the weight of the pulleys), what is the number of the pulleys?

30. In the system of pulleys in which each string is attached to the weight, there are three moveable pulleys of weights w_1, w_2, w_3, beginning with the lowest, and the force P then balances a weight W; when the first and second pulleys are interchanged then a force P' balances W; shew that $P - P' = \frac{4}{15}(w_1 - w_2)$.

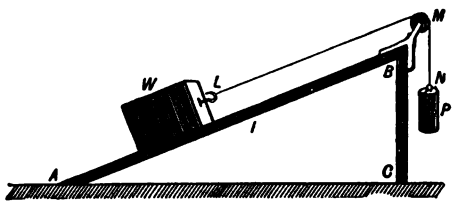
31. There is one system of pulleys in which the weight of the pulleys increases the mechanical advantage; which is the system?

THE SMOOTH INCLINED PLANE.

236. An **Inclined Plane** is a rigid plane making a finite angle with the horizon.

In what follows we consider the inclined plane to be a machine in which all the forces acting are in one plane; this plane is a vertical plane perpendicular to the intersection of the inclined plane with the horizon, and therefore cutting the inclined plane in a *line of greatest slope.*

237. The representation on paper of an inclined plane is the section made with it by the vertical plane in which all the forces are supposed to act.



Thus in the figure AB is a line of greatest slope of the plane.

An inclined plane is often supposed to be of a finite **length** AB, in which case the lines BC and AC being drawn vertically and horizontally through B and A, BC is called the **height** and AC is called the **base.**

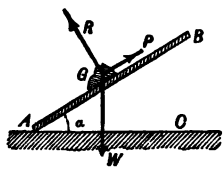
238. In the present Chapter we shall suppose the inclined plane perfectly *smooth;* so that it can only apply to a body a pressure in the direction perpendicular to its surface. [See Art. 255.]

239. *To find the relation between the* Power *and the* Weight, *when the Power is parallel to the plane.*

Let α be the inclination of the plane to the horizon.

The Weight W is the weight of a body placed on the plane acting at G its Centre of Gravity.

The Power P is a force applied to the body by a string or otherwise up the plane parallel to it.



Let R be the pressure of the plane on the body.

R is therefore a force perpendicular to the plane acting on the body through its Centre of Gravity.

These three forces W, P, R which act on the body are in equilibrium.

Taking the sum of the resolutes along the plane, of all the forces,

we have $W \sin \alpha - P = 0$ ..... (i)*.

Taking the sum of the resolutes perpendicular to the plane, of all the forces,

we have $W \cos \alpha - R = 0$ .....(ii).

Example. The height of an inclined plane is to its base as 5 : 12; *find the* Power P *parallel to the plane which will support a Weight of* 13 cwt.

Let the height CB of the plane $= 5$ m.; then the base $AC = 12$ m. and the length AB is $\sqrt{(12^2 + 5^2)}$ m., that is 13 m. [See fig. on p. 180.]

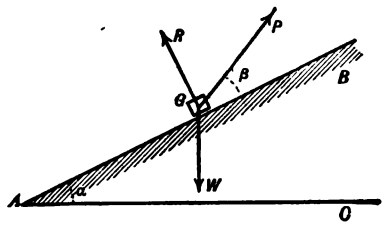
Now P is equal to the resolute of W along the plane; that is,

$$P = W \sin \alpha = \tfrac{5}{13} W;$$

$$\therefore P = \tfrac{5}{13} \text{ of } 13 \text{ cwt.} = \underline{5 \text{ cwt.}}$$

* It follows from this that when a mass m lbs. is placed on a smooth inclined plane it moves along the plane downwards under the action of $mg \sin \alpha$ poundals; i.e. with the acceleration $g \sin \alpha$ celos down the plane.

240. PROP. *To find the relation between the* Power *and the* Weight, *when the Power makes any given angle* β *with the inclined plane.*



Let W be the weight of the body on the plane; let P be the Power; and let R be the pressure of the plane on the body.

Let α be the inclination of the plane to the horizon.

Then the forces W, P, R are in equilibrium, therefore the sum of their resolutes parallel to the plane is zero.

$$\therefore \quad W \sin \alpha - P \cos \beta = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

The sum of their resolutes perpendicular to the plane is zero.

$$\therefore \quad W \cos \alpha - R - P \sin \beta = 0 \ldots\ldots\ldots \text{(ii)}.$$

Hence $$P = W \frac{\sin \alpha}{\cos \beta},$$

and
$$R = W \cos \alpha - P \sin \beta$$
$$= W \cos \alpha - W \frac{\sin \alpha \sin \beta}{\cos \beta}$$
$$= W \frac{\cos (\alpha + \beta)}{\cos \beta}.$$

241. NOTE. **For instance**; Let P act horizontally, then $\beta = -\alpha$, and
$$P = W \tan \alpha,$$
$$R = W \sec \alpha.$$

EXAMPLES. XLIV.

1. Prove that when a body is kept in equilibrium on a smooth inclined plane and the power acts along the plane, the Power is to the Weight as the height of the plane is to the length.

2. When a body is kept in equilibrium on an inclined plane and the power is horizontal, then the Power is to the Weight as the height of the plane is to the base.

3. If the inclination to the horizon of a plane be 60°, find the force which acting horizontally would support a weight of 12 lbs.

4. A weight of 12 lbs. resting on a smooth plane at an angle of 30° to the horizon is fastened to a cord which passes up the plane and over the top; find what weight must be attached to the cord to preserve equilibrium.

5. Find the inclination of a plane on which a horizontal force will support a weight equal to itself.

6. Find the inclination of a plane on which a power parallel to the plane will support double its own weight.

7. The angle of an inclined plane is 30°, and a force P acting horizontally keeps a weight in equilibrium; if P acts in a direction making the angle 30° with the plane and above it, shew that it will still maintain equilibrium and that the pressure on the plane will be reduced one half.

8. AB, AC are two smooth planes inclined to the horizon at 60° and 30° respectively; a weight P on AB and a weight Q on AC are connected by a string which passes over a pulley at A. If P and Q are in equilibrium, what is the ratio of their weights?

9. An inclined plane makes the angle 30° with the horizon, and a weight W is supported on it by a force P such that $2P^2 = W^2$; in what direction does P act?

10. An inclined plane makes the angle 45° with the horizon and $2W^2 = 3P^2$; in what direction does P act?

11. The Powers, which when acting horizontally and parallel to the plane respectively will support a given weight, are in the ratio of 2 to 1; what is the angle of the plane?

12. A body weighing 9 lbs. is in equilibrium upon an inclined plane under the action of a horizontal force of $3\sqrt{3}$ lbs.; what is the inclination of the plane and the pressure on it?

13. A heavy weight is fastened by means of a string to a fixed point, and rests on a smooth plane inclined to the horizon at the angle 30°, the direction of the string making the angle 60° with the horizon; shew that the tension of the string is equal to the pressure on the plane.

14. $W : P : R = \sqrt{3} : 1 : 1$; find the inclination of the plane and the direction of P.

15. A weight $2P$ is kept in equilibrium on an inclined plane by a horizontal force P and a force P acting parallel to the plane; find the ratio of the base of the plane to the height and the pressure on the plane.

16. Two planes AB, AC having a common height are inclined to the horizon at angles α and β respectively. Two weights, one on each plane, are kept in equilibrium by a string attached to the weights and passing over A. Find the ratio of the weights.

17. A power P acting parallel to the plane can support W, and acting horizontally can support W'; prove that

$$P^2 = W^2 - W'^2.$$

18. A railway train of 160 tons is supported on an incline of 1 in 80 by means of a rope parallel to the plane; find the tension of the rope.

19. If R be the pressure when P acts horizontally, and R' when it acts parallel to the plane, then $RR' = W^2$.

20. If a horizontal power P supports W, and P' parallel to the plane supports W, then $\frac{1}{P'^2} - \frac{1}{P^2} = \frac{1}{W^2}$.

21. A force P acting along a given plane can support a weight W_1, and acting horizontally can support a weight W_2; prove that

$$P^2 = W_1^2 - W_2^2.$$

22. W can be supported on a given smooth inclined plane by the force P along the plane, or by the horizontal force $2P$; if R_1, R_2 be the pressure on the plane in these two cases, prove that

$$R_1 R_2 = \tfrac{4}{3} P^2.$$

23. For a given value of P and a given value of W shew that there are generally two inclinations in which P may act so as to support W on a given smooth inclined plane.

24. What weight can be supported on a smooth inclined plane of angle $45°$ by a horizontal force of 3 lbs. and a force of 4 lbs. parallel to the plane acting together?

25. A mass of 10 lbs. is placed on a smooth inclined plane of angle $30°$; how long will it take to slide 32 feet down the plane from rest?

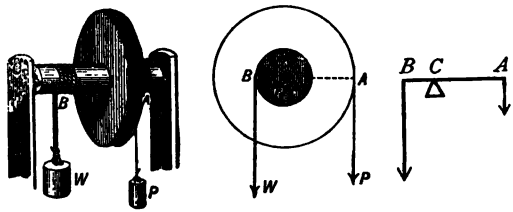
26. A particle slides down a smooth inclined plane of angle $30°$; shew that it takes twice as long to descend a given distance as a particle falling freely under gravity.

27. A mass is drawn up a smooth inclined plane of angle $30°$ by an equal mass which descends vertically; the masses are connected by a light string which passes over a smooth pulley at the top of the plane; prove that the acceleration is $\frac{1}{4}g$ celos.

THE WHEEL AND AXLE.

242. In this machine the **axle** is a *cylinder* with a fixed *axis*, and the **wheel** is fixed to the cylinder and turns about the same axis.

Thus in the figure, a rope coiled round the cylinder B supports a weight W; and a cord coiled round the wheel in the opposite direction supports a weight or force P.



243. The forces acting on the rigid body composing the wheel and axle are not all in one plane, nor are they symmetrical about any one plane; therefore their theory is not properly within the scope of this work.

We shall assume however that the force applied by the rope to the axle at B acts just as if it were in the plane of the wheel; as in the second figure.

With this assumption the theory is simply that of a rigid body acted on by forces in one plane with one point (the centre of the wheel) fixed.

244. PROP. *To find the relation between the* Power *and the* Weight *in the Wheel and Axle.*

Let the plane of the paper be the central plane of the wheel and let C be the centre of the circle in which the wheel cuts that plane.

Let the cord passing round the wheel touch this circle at A, and let the rope passing round the axle touch the axle in B.

Let R be the radius of the wheel and r the radius of the axle.

Then we have by Art. 196, (the only necessary condition of equilibrium, since the point C is fixed)

$$W \times r = P \times R,$$

or $W \times$ radius of axle $= P \times$ radius of wheel.

245. It will be seen that the practical effect of the arrangement of the wheel and axle is, that it is a kind of continuous lever of the first class—the axis is the fulcrum; the radius of the wheel is the longer arm; the radius of the axle is the shorter arm of the lever.

The mechanical advantage [Art. 212] of this machine is

$$\frac{\text{radius of wheel}}{\text{radius of axle}}$$

and this may theoretically be made as large as we please; but practically its magnitude is limited (i) by the fact that a very large wheel is costly and unwieldy, (ii) a very small axle cannot be made of sufficient strength to sustain any very great strain. See Examples L. 10, 11.

EXAMPLES. XLV.

1. Shew that the power may be applied to the wheel in any direction provided its magnitude is unchanged and provided it is kept always perpendicular to the axis.

2. Shew that if the weight is n times the power in the wheel and axle, then the cords round the wheel must be unwound n feet in order to wind the rope 1 foot round the axle.

3. Four sailors each exerting a force capable of supporting 116 lbs., lift an anchor by means of a capstan, whose radius is 1 ft. 2 in. and whose spokes are 8 feet long (measured from the axis); what is the weight of the anchor?

4. To raise a block of stone of 1 ton a single pulley is used at the top and a windlass worked by two men at the bottom; if the crank of the windlass is 2 feet and the radius of the axle 4 inches what force perpendicular to the cranks must each man exert at the least? Does it make any difference if the direction of the chain passing from the pulley to the windlass is not vertical?

5. If the string from an axle of radius 3 inches passes round a moveable pulley and has its end fixed to a beam above (that part of the string which does not touch the axle or pulley being vertical) and if a weight of 2 tons (fastened to the moveable pulley) is supported by a power of 1 cwt. (applied to the wheel), find the radius of the wheel.

6. The radius of the wheel is 6 feet, the radius of the axle 6 inches; a weight of 8 lbs. is fastened to a rope coiled round the axle and a weight of 4 lbs. is fastened to a rope nailed to the rim of the wheel; find the position of equilibrium.

7. In what direction must the power act in order that the pressure on the axle may be (i) the least possible, (ii) the greatest possible?

8. If the axis of the axle do not coincide with the centre of the wheel, shew that the sum of the weights which are the greatest and least with which a given power can be in equilibrium is double the weight which this power would support if the axis of the same wheel did coincide with the axis of the axle.

9. A wheel is made square, the side of the square being 2 ft.; the axle is round and its radius is 3 inches; find the position which the machine will assume when the weight is 5 lbs. and the power 1 lb.

10. To the same axis are fixed a wheel of radius R, a smaller wheel of radius r, and an axle of radius r_2. A rope is coiled round the axle and after passing over a single moveable pulley is coiled the other way round the smaller wheel. The power P acts in the proper direction on a string coiled round the larger wheel and the weight W is attached to the moveable pulley; prove that

$$W = P \times \frac{2R}{r_1 - r_2}.$$

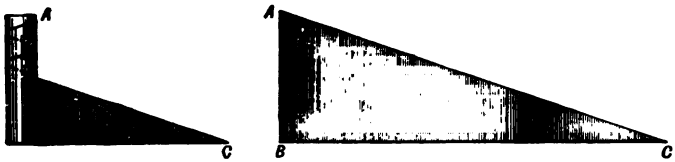
The above arrangement is called the **Differential Wheel and Axle.**

11. Shew that the mechanical advantage of the Differential Wheel and Axle can be made as large as we please without unduly either enlarging the wheel or diminishing the axle.

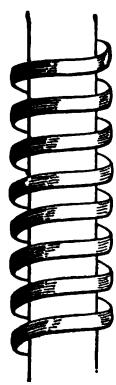
THE SCREW.

246. The **screw** may be described as an *inclined plane* wound round a cylinder.

Take a piece of paper in the form of a right-angled triangle *ABC* as in the figure and wind it round a cylindrical pencil; the slant side of the triangle will give the trace of a screw on the surface of the pencil.



247. A screw is acted on by a couple which tends to make it turn about its axis. The thread of the screw is fitted into a counter part so that as the screw turns, the cylinder on which the screw is traced moves in the direction of its axis.



If we suppose the screw perfectly smooth, and assume the principle of Work (viz. that what is lost in force is gained in distance moved), we can find the relation between P and W.

248. Let a force P be applied at the end of a lever, whose arm is l inches long, so as to cause the screw to tend to turn; and suppose that this tendency to turn is counteracted by a pressure W against the head of the screw in the line of its axis, opposing its consequent tendency to move parallel to the axis.

Now suppose the distance between the threads of the screw to be n inches; then for each complete turn of the screw the axis would move forward n inches;

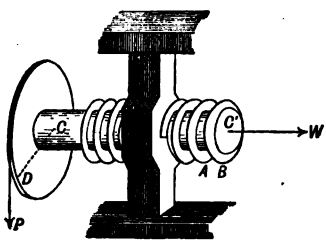
also, if the axis were vertical and the pressure against the screw were caused by a weight W resting on it, each complete turn of the screw would raise W a height of n inches.

But in doing this, the force P, applied at right angles to the lever, would have to work through the whole circumference of the circle of radius l inches; that is, it would have to work through the distance $2\pi l$ inches.

Therefore, by the Principle of the Indestructibility of Work, viz. that in a machine in which the constraints are all *smooth*, the work done on the machine is equal to the work done by the machine, we have

$$P \times 2\pi l = W \times n.$$

In practice a great deal of this work $P2\pi l$, even under the most favourable circumstances, is consumed in overcoming the friction of the screw; so that in practice Wn is considerably less than $2P\pi l$.



249. The figure here given shews a combination of wheel and screw which will help the student to understand, that while W works through the distance between two threads of the screw, the distance which P works through is the circumference of the wheel.

Suppose a cord coiled round the wheel, and a weight of P lbs. attached to it; for each complete turn of the screw, this weight P would descend a distance equal to the circumference of the wheel; that is, it would descend about twice $\frac{22}{7}$ths of the radius ($2\pi l$).

In the figure $CD=l$, $AB=n$.

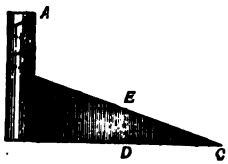
The work done by P is thus $2\pi l \times P$.

The work done on $W = W \times$ distance between two threads $= n \times W$.

By the principle of the indestructibility of work [see Chapter XXIII] (since the pressures of the constraints do no work) we have

$$2\pi l \times P = n \times W.$$

250. The **pitch** of a screw is the angle of inclination of the inclined plane which when wound round a cylinder forms the screw. Thus in the figure the angle ECD is the



pitch. If DC be the circumference of the cylinder then DE is the distance between two consecutive threads, and

$$DE = DC \tan i,$$

where i is the *pitch* of the screw.

Hence, the distance between the threads of a screw = the circumference of the screw × tan i.

Example. What must be the length of the power arm of a screw having 6 threads to an inch that its mechanical efficiency may be 216?

The mechanical efficiency of a machine is the ratio of the Weight to the Power.

In the screw $P \times 2\pi$ times the power arm $= W \times$ distance between the threads;

$\therefore$ 2π times the power arm $= 216 \times \frac{1}{6}$ of an inch.

Take π as $= \frac{22}{7}$; then

$$\text{the power arm} = \frac{216 \times 7}{6 \times 44} \text{ inches} = 5{\cdot}73\ldots\text{inches} = \underline{5\tfrac{3}{4} \text{ inches.}}$$

EXAMPLES. XLVI.

1. In a Screw the circumference of the circle described by the extremity of the power arm is 2 feet and the distance between the threads $\frac{1}{2}$ in.; what weight will a power of 28 lbs. support?

2. If in a Screw the power arm is 2 ft. long and the distance between the threads 1 inch, what power will support 1 ton?

3. If the mechanical advantage of a Screw is 1000 and the length of the power arm is 25 inches, what is the distance between the threads?

4. If the extremity of the power arm describes a circle of 10 ft. and a force of 1 lb. supports 1 ton, what is the distance between the threads of the Screw?

CHAPTER XIX.

FRICTION.

251. We propose in this Chapter to consider the nature of the stress set up between the surfaces of two separate *rigid* bodies which press the one on the other.

252. We shall confine our attention to the case of a fixed plane and a body placed upon it.

253. When a *rigid* body has one of its surfaces in contact with a fixed rigid plane, then, so long as the surface remains in contact with the plane, the only motion of which the body is capable is in some direction parallel to the plane.

For, whatever may be the forces acting upon the body, the plane, being rigid, can and does exert upon it whatever force is necessary to prevent motion in the direction perpendicular to the plane.

This force, applied by the plane to the body in the direction perpendicular to its surface, is equal and opposite to the perpendicular **pressure** of the body on the plane.

254. If there were a rigid substance whose surface could be made *perfectly* smooth, then the pressure applied by such a surface to any other surface pressing against it would be *exactly in the direction at right angles to the smooth surface.*

Although no such substance is known, yet it is often convenient to imagine such a substance for the purposes of theoretical statics. For the surfaces of some substances can be rendered very much more smooth than those of others, and hence we obtain results which are approximately true for surfaces which are comparatively smooth.

Moreover, for the purposes of explanation, we at first make our problems as simple as possible; advancing from the simple problems of theory to the more complex problems of practical mechanics.

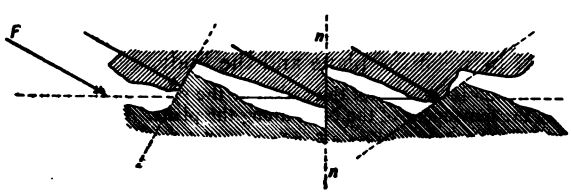
Accordingly we use the words *smooth* and *rough* in the following technical sense.

255. DEF. A surface is said to be **smooth** which is understood to be incapable of exerting pressure on any other body in contact with it *except in the direction perpendicular to itself.*

256. DEF. A surface is said to be **rough** which is capable of exerting pressure on a body in contact with it *in other directions* besides that perpendicular to itself.

It will be seen that a smooth surface may be said to be one which offers no resistance to the motion along it of a body which is pressing against it.

A rough surface does offer some resistance to the motion along it of a body pressing against it.



For all actual surfaces, even those that appear to be perfectly plane, would if sufficiently magnified be seen to consist of minute projections and depressions as in the Figure. These always interlock more or less with another surface in contact with it, and oppose resistance to a force such as the horizontal component of F which tends to slide one surface over the other.

Sometimes the projections interlock so that the sliding cannot take place without scratching as at n, n in the figure.

257. DEF. A **perfectly rough** surface is one which is capable of offering whatever resistance may be necessary to prevent the motion along it of a body pressing against it.

Thus a *perfectly* rough surface is a theoretical surface along which bodies pressing on it cannot move.

258. It is as we have said impossible to get a surface *smooth* as defined above. It must however be understood that when we speak of a 'rough' plane surface we mean a surface which has been made as nearly smooth as the nature of the substance will permit.

For instance, if the material is of *metal* it is understood that the surface has been carefully polished; if of *wood*, that it has been planed; if of *stone*, that it has been rubbed down, and brought to as smooth a surface as the particular kind of stone will allow. Its *surface* will then depend only on the material; it will be smooth to the eye and touch, but not *statically* smooth.

259. Consider now the action exerted by a fixed 'rough' plane upon a body which presses against it.

Since the plane is rough, the force applied to the body is not necessarily perpendicular to the plane.

We resolve this force into two rectangular components, one at right angles to the plane, the other along the plane.

The component force at right angles to the plane is called the **pressure** between the plane and the body.

The component force along the plane is called the **friction** between the plane and the body.

The friction is the part of the action of the plane on the body which prevents the movement of the body along the plane.

When a body is at rest on a fixed plane the action of the plane upon it is equal and opposite to the resultant of all the external forces acting on the body. The friction is therefore equal and opposite to the part of the external forces which tends to cause motion of the body on the plane. Hence we have the following definition.

260. DEF. When a body rests on a rough plane and forces act on the body tending to cause it to slide along the plane, a force is called into play which acts on the body in the direction contrary to that in which it tends to move and tends to prevent motion.

This force is called **Friction.**

261. The **General Laws of Friction** are

I. When the surfaces of two bodies are in contact and at rest relatively to each other friction always acts upon each body in the direction exactly opposite to that in which the body tends to move along the surface on which it presses.

II. When there is no relative motion, the amount of friction which is called into action is just as much as is necessary to prevent relative motion—and no more.

Example i. *A mass of weight* W *rests on a rough horizontal plane; it is pulled in a certain direction by a horizontal force of* P lbs.; *the mass does not move.* Why is this?

The mass is under the action of the following forces:

(i) its own weight downwards,

(ii) the pressure of the plane on it upwards,

(iii) the force P horizontally,

(iv) the force of friction, acting between the surface of the mass and of the plane; this force is horizontal and must be equal to the only other force acting, namely P lbs., and must act in exactly the opposite direction.

If the plane in the present case were perfectly smooth, in other words, if we suppose friction not to exist, it is clear that the force P would cause the mass to move in its own direction.

Example ii. *A heavy mass resting on a horizontal plane is acted on by a force of* 4 lbs. *towards the North and a force of* 3 lbs. *towards the West. The mass is kept at rest by the friction between itself and the plane. What is the force of friction called into play?*

The resultant of the two horizontal forces is a force of 5 lbs. acting in a direction somewhere about N.W.N.

The force of friction must therefore be a force of 5 lbs. acting exactly opposite to this resultant.

262. A little thought will enable the student to understand how great a part friction plays in practical mechanics.

It is friction which keeps the furniture of a room in its place; which enables us to walk and move at will. It is interesting to try and realize for a moment how different the world would be if this force of friction ceased to act. The student will find that very few problems would then be *statical* problems. Almost every problem which could be proposed in mechanics would necessarily involve motion.

EXAMPLES. XLVII.

1. A body rests on a horizontal plane and is acted on by the force of 10 lbs. in the direction making the angle 60° with the plane. What amount of friction is called into play?

2. A body rests on a horizontal plane and is acted on by the force of 40 lbs. in the direction making the angle 45° with the plane. What friction is acting on the body?

3. A body weighing 40 lbs. rests on an inclined plane whose inclination to the horizon is 30°. What amount of friction is acting between the body and the plane?

4. What will the friction be if the plane in Question 3 be inclined at the angle 45° to the horizon?

5. A body weighing 50 lbs. rests on an inclined plane whose angle is 30° and a force of 10 lbs. is acting upon it up the plane; what friction is there?

6. If the body in Question 5 weighs 112 lbs. and the force up the plane is 112 lbs.; what friction is called into play?

7. A body weighing 1 ton is placed on an inclined plane of inclination 60° and is acted on by a force of 1 ton up the plane; what friction is acting on it?

8. A body rests on a horizontal plane and is pulled horizontally by two men who exert forces of 30 lbs. towards the North and 40 lbs. towards the East respectively; what friction is acting on the body?

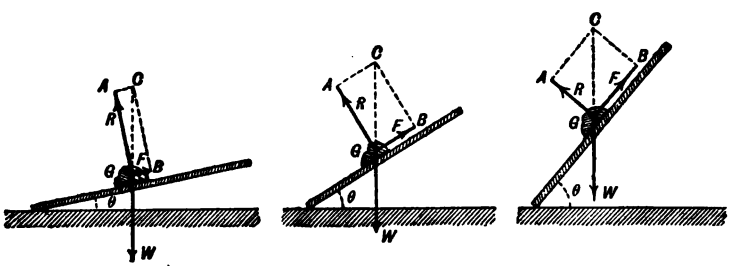
9. A mass of 1 cwt. lies on an inclined plane of angle 30° and two men push against it, one up the plane with the force 56 lbs. and the other down the plane with the force 84 lbs. What friction is called into play?

LIMITING FRICTION.

263. It is most important that the student should realize that in any particular case in which there is no motion, the amount of friction actually called into play is only just so much as is necessary to prevent sliding motion.

In most cases however if the force tending to cause sliding motion be increased, it will be found that there is a **limit** to the amount of friction which can be called into play. So that when the force necessary to prevent sliding motion is greater than a certain amount, the surfaces slide the one on the other.

264. *Illustration.* Suppose we take a plane such as the surface of a piece of board. Let the board have a hinge so that it can be tilted up at different inclinations to the horizon and let its surface be made as smooth as the nature of the substance of which it is made will permit.



Now take a rigid body such as a lump of iron; let its face be made as smooth as possible and let it be placed without restraint on the plane.

First let the plane be horizontal.

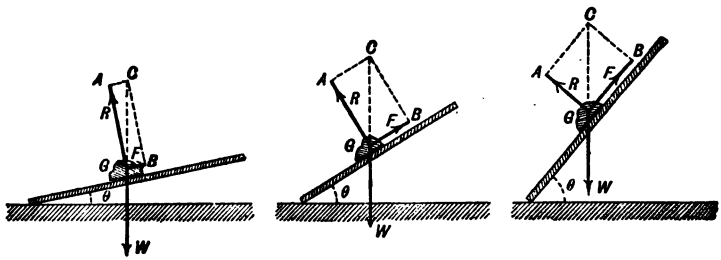
The body will be at rest; as the only forces acting upon it are its weight downwards and an equal pressure of the plane on the mass upwards.

Next let the plane be tilted up so as to make a small angle (θ) with the horizon; and suppose that the body still remains at rest.

The forces now acting upon the body are its weight vertically downwards and the action of the plane on it; these two forces must be equal and opposite. Hence the action of the plane on the body must still be vertical; that is, the action is no longer perpendicular to the plane.

We resolve this action into two rectangular components, (i) perpendicular to the plane, which we call the **pressure** (R), (ii) along the plane which we call the **friction** (F).

These two (i) pressure (R), and (ii) friction (F), are the



rectangular components of a constant vertical force upwards, equal to (W) the weight of the body.

It will be seen from the figure that as the angle of inclination θ of the plane is increased R diminishes and F increases.

For $R = W\cos\theta$, $F = W\sin\theta$, and $\therefore \dfrac{F}{R} = \dfrac{W\sin\theta}{W\cos\theta}$ $= \tan\theta$.

Hence with a small inclination of the plane the ratio $\dfrac{F}{R}$ is small, but this ratio increases with θ and it can be made as great as we please (for $\tan 90^\circ = \infty$).

Now it will be found (if the experiment be performed carefully) that with a plane made of any chosen kind of material, say box-wood, and a body of any other chosen material, say soft iron, motion will always begin when (θ) the inclination of the plane reaches a certain magnitude (α). That is, when the ratio of $\dfrac{F}{R}$ attains a certain maximum value ($\tan\alpha$).

Experiments indicate that provided the surfaces in contact have been made as smooth as possible, this angle α at which motion begins, is for most practical purposes, independent (i) of the weight of the mass, and therefore of the magnitude of the pressure R, (ii) of the size of the surface in contact.

Thus if masses of the *same materials* but of different sizes and shapes be used for repetitions of the experiment, the angle α will be found to be approximately the same in every case.

265. It is from the results of a careful series of experiments such as the one described above that we obtain the following

LAWS OF LIMITING FRICTION.

I. When two bodies whose surfaces are in contact have a tendency to relative motion it is found that—when the surfaces have been made as smooth as possible—the amount of friction which can be called into play *cannot exceed a certain limiting value.*

II. This limiting value is a certain fraction of the normal pressure.

This fraction *depends on* the nature of the bodies in contact, and is called the **coefficient of friction** of those bodies.

Thus the coefficient of friction for soft iron and brass is about ·17.

This fraction *is independent of* the size and shape of the surfaces in contact.

It is to be noticed however that the surface must be large enough to prevent actual scratching of the surfaces.

III. When motion actually is taking place the above laws are still approximately true as to the friction between sliding bodies; but the fraction $\frac{F}{R}$ is rather less in each particular case when the bodies are sliding than when just *on the point* of motion.

It seems as if the bodies when at rest have time to interlock their projections more completely than when in motion the one over the other; so that it takes rather more force to start them than to keep them moving uniformly.

SOME COEFFICIENTS OF FRICTION.

From MORIN.

Substances in Contact.	Coef.	Angle.
Oak on oak, all the fibres parallel to the motion	·48	25°.38′
,, ,, moving fibres on end, resting fibres parallel to the motion	·19	10°.46′
Wrought-iron on brass	·17	9°.39′
Steel on cast-iron	·20	11°.19′
Steel on polished glass	·11	6°.17′
Polished marble on polished marble	·16	9°. 6′
Polished marble on common birch	·44	23°.45′
Common birch on common birch	·64	32°.38′
Wrought iron on oak (fibres parallel to motion)	·62	31°.47′
,, ,, elm ,, ,, ,,	·25	14°. 3′

The above are quoted from a very full list given on page 373 of Trautwine's *Engineer's Pocket-book*. Ed. 1888.

266. NOTE. In the above table the coefficient of Limiting Friction is the tangent of the angle at which the body will begin to slide when on an inclined plane as in Art. 212. This will be clear if the student will notice that in the experiment the action of the plane on the body is always *vertical*, and therefore this action makes the same angle with the perpendicular to the plane which the plane makes with the horizon. It is usual to use the letter μ to denote a coefficient of friction. Thus,

$$F = \mu R.$$

But, if α is the limiting angle,

$$F = R \tan \alpha;$$

$$\therefore \mu = \tan \alpha.$$

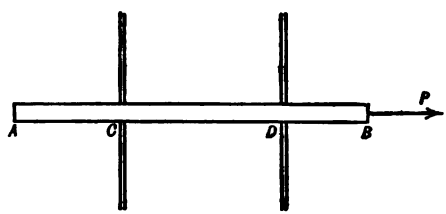
Example i. *An iron girder weighing* 1 ton *lies on* horizontal *steel rails; what horizontal force must a horse apply to the girder to make it move forward along the rails and in the direction of its length, if the coefficient of friction for iron and steel is* ·175?

The whole pressure of the girder on the rails is 1 ton = 2240 lbs.; this pressure is distributed over the surface of that part of the girder in contact with the steel rails.

When a horizontal force has been applied to the girder so that it is on the point of motion, then at every part of the surfaces at which there is pressure there is a horizontal force of friction opposite to the direction in which there is a tendency to motion. This friction is in all cases ·175 of the pressure.

For the direction in which the girder is to be pulled is in the direction of its length, so that if any of the points of contact be on the point of motion, *every* point must be on the point of motion.

[Thus if AB be the girder, resting on two horizontal steel rails at C and D, the pressures on the rails at C and D may be unequal; suppose them to be R_1 and R_2.



Then $R_1 + R_2 = 1$ ton,

and the limiting friction due to $R_1 = ·175 \times R_1$ and the limiting friction due to $R_2 = ·175 \times R_2$. So that the girder will be on the point of motion when a force $·175R_1 + ·175R_2$ is applied to it in the direction AB, that is ·175 times 1 ton.]

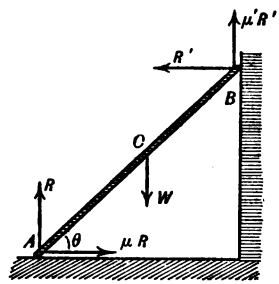
Therefore the whole amount of the friction is ·175 of the whole amount of the pressure.

$\therefore$ the limiting friction $= ·175 \times 2240$ lbs.

$= 392$ lbs.

Therefore if a horizontal force of any magnitude exceeding 392 lbs. be applied to the girder in the direction of its length it will begin to move in the direction of the force.

Example ii. *A uniform ladder rests with one end on a horizontal stone pavement, the other leaning against a vertical brick wall; find the limiting position of equilibrium; the coefficients of friction being respectively* μ *and* μ'.



With the notation of the figure, in which the ladder AB is supposed to be on the point of slipping down, we get by [Art. 91] considering the conditions for the equilibrium of the ladder, and

I. resolving horizontally,

II. resolving vertically, and

III. taking moments about C we have

I. $$R' - \mu R = 0.$$

II. $$R + \mu' R' - W = 0.$$

III. $$R\cos\theta - \mu R\sin\theta - R'\sin\theta - \mu' R'\cos\theta = 0;$$

$$\therefore \quad \frac{R}{R'} = \frac{\sin\theta + \mu'\cos\theta}{\cos\theta - \mu\sin\theta}$$

$$= \frac{1}{\mu} \text{ [from I.]},$$

$$\therefore \quad \mu\sin\theta + \mu\mu'\cos\theta = \cos\theta - \mu\sin\theta;$$

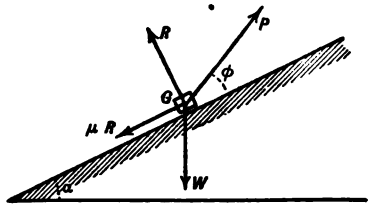
$$\therefore \quad \tan\theta = \frac{1 - \mu\mu'}{2\mu}.$$

That is, the angle of inclination of the ladder to the horizon must not be less than the angle whose tangent is $\frac{1-\mu\mu'}{2\mu}$.

For instance if $\mu = \mu' = \frac{1}{2}$;

$$\tan\theta = \frac{3}{4}.$$

Example iii. *Let a mass* W *resting on a rough inclined plane, be acted on by a force* P *making an angle* ϕ *with the plane, as in the figure. Find the relation between* P, W, ϕ.



With the notation of the figure, **first** we shall find the relation between P, W, ϕ when the mass is on the point of moving *up* the plane.

In this case, the pressure on the plane being R the friction called into play is μR and acts *down* the plane as drawn.

We get by [Art. 91] considering that the mass is in equilibrium, and resolving I. along, and II. perpendicular to the plane

I. $W\sin\alpha+\mu R-P\cos\phi=0.$

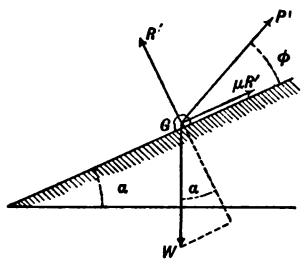
II. $R+P\sin\phi-W\cos\alpha=0.$

Multiplying II. by μ and subtracting this from I. we have

$$W\sin\alpha-P\cos\phi-\mu P\sin\phi+\mu W\cos\alpha=0,$$

or
$$W(\sin\alpha+\mu\cos\alpha)=P(\cos\phi+\mu\sin\phi).$$

This is the equation which will give the magnitude of the force necessary to cause the mass to be on the point of motion up the plane when W, α, μ and ϕ are given.



Next, supposing the mass on the point of moving down the plane, the force μR must be reversed as in the figure; the effect of this will be to change the *sign* of μ in the above equations. Thus

I. $W\sin\alpha-\mu R'-P'\cos\phi=0.$

II. $R'+P'\sin\phi-W\cos\alpha=0.$

Whence
$$P'=W\times\frac{\cos\alpha-\mu\cos\alpha}{\cos\phi-\mu\sin\phi}.$$

EXAMPLES. XLVIII.

1. A weight placed on a rough inclined plane whose inclination is 30° is just on the point of sliding down; what is the coefficient of friction?

2. A mass of iron weighing 1 ton placed on a horizontal plane made of brass; what horizontal force is necessary to move it?

3. A body placed on a rough inclined plane whose angle is α is just on the point of motion; what horizontal force is necessary to move a body of 1 cwt. of the same material placed on the plane when horizontal?

4. A body placed on an inclined plane of angle 30° is just on the point of moving upwards when acted on by a horizontal force equal to its own weight; find the coefficient of friction.

5. A body placed on a horizontal plane requires a horizontal force equal to its weight multiplied by $\sqrt{3}$ to cause it to be on the point of motion; at what angle must the plane be tilted that the body may then be on the point of motion?

6. A body placed on a rough horizontal plane is on the point of motion when acted on by a force equal to its own weight inclined to the horizon at an angle of 60°; find the coefficient of friction.

7. If the inclination to the horizon of the force in Question **6** were 45°, what would be the coefficient of friction?

8. A mass of 1 lb. placed on an inclined plane is fastened to the plane by a thread which is parallel to the plane; the coefficient of friction being ·2, what must be the least tension of the string when the inclination of the plane is 45°?

9. A heavy beam rests with one end on a horizontal pavement and the other in contact with a vertical wall; if the coefficient of friction of the ground and wall be equal to ·3 and the beam is uniform, find the inclination of the beam to the horizon when it is on the point of slipping down.

10. A ladder 20 ft. long without weight rests with one end on the ground which is rough ($\mu = \cdot 4$) and the other against a vertical wall which is smooth and is inclined to the horizon at an angle of 45°; a man commences to walk up the ladder; how far will he go before the ladder slips?

11. Shew that if in Question **10** the ground is smooth and the wall rough, the ladder will slip wherever it is placed directly the man attempts to ascend it.

12. A uniform ladder 20 ft. long weighing 56 lbs. rests with one end on the ground which is rough (coefficient of friction $= \mu$) and the other against a smooth vertical wall; find the least angle to the horizon at which it can rest; and find how far a body of 1 cwt. could walk up the ladder when at that angle, without causing it to slip.

13. It is said that the force required to keep a train moving uniformly on an incline of 1 in 80 is four times the force necessary on horizontal rails; assuming that the friction varies as the normal pressure, find the coefficient of friction.

14. Taking the same coefficient of friction as in Question **13**, find how many times the force required on a horizontal rail is required on an incline of 1 in 60.

SECTION VI.

CENTRIPETAL FORCE.

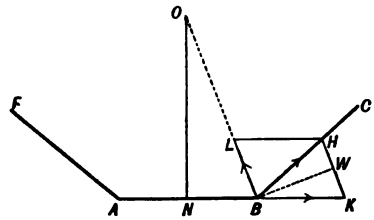
CHAPTER XX.

CIRCULAR MOTION.

267. PROP. *A point moves with constant speed* v velos *along the perimeter of a regular polygon* ABC... *of* n *sides inscribed in a circle of radius* r ft.; *prove that at each angle the point receives an additional velocity towards the centre of the circle; and that the* rate of additional velocity per second *is* $\frac{v^2}{r}$.

Let O be the centre of the circle.

Let $AB = s$ ft.; let τ seconds be the interval occupied in going from A to B; then, $s = v\tau$.



Draw ON bisecting AB at right angles.

Produce AB to K so that BK represents the velocity v velos along AB;

let BH represent the velocity v velos along BC; then

$$BH = BK.$$

Complete the parallelogram $BKHL$.

Draw BW bisecting HK at right angles.

Then, since $BK = BH$, the angle $BKH = BHK = HBL$.

But the angle $ABL = BKH$;

therefore the angle $ABL = HBL$.

Therefore BL produced passes through the centre O.

Now the velocity represented by BH is the resultant of the velocities BK, BL. That is, *the velocity* BL *when added to the velocity* BK *changes it into the velocity* BH.

In other words, when a point having v velos along AB changes its velocity to v velos along BC, it receives the additional velocity w velos represented by BL. Also in moving along the perimeter of the polygon the point changes its velocity at intervals of τ seconds. Therefore its rate of additional velocity per second is w/τ velos per second.

Now $\frac{w}{v} = \frac{KH}{BK} = \frac{2KW}{BK} = \frac{2BN}{OB} = \frac{s}{r}$; $\therefore w = v \cdot \frac{s}{r}$;

also, $$\frac{1}{\tau} = \frac{v}{s}.$$

Therefore $$\frac{w}{\tau} = v \cdot \frac{s}{r} \times \frac{v}{s} = \frac{v^2}{r}.$$ That is,

the rate of additional velocity is $\frac{v^2}{r}$ velos per second. Q.E.D.

268. PROP. *A point moves with the constant speed* v velos *along the circumference of a circle of* radius r; *prove that this point has the* **constant acceleration** v^2/r celos **towards the centre of the circle.**

We consider a circle to be the curve which a regular polygon approaches as its limit, when the number of its sides is infinitely increased.

When the polygon in Art. 267 becomes a circle the additional velocity at each angular point is infinitely diminished, the number of angular points being infinitely increased.

Hence, instead of a series of sudden increases of velocity we have, for the circle, a *continuous* growth of velocity, always in the direction of the centre, at the rate v^2/r velos per second; that is, there is the acceleration v^2/r celos towards the centre.

N.B. Neither in the polygon nor in the circle does this increase of velocity alter the *speed*. It alters the velocity as regards *direction* only.

269. PROP. *A particle of mass* m lbs. *is describing a circle of radius* r ft. *with* v velos; *prove that it is acted on by a force* $\frac{mv^2}{r}$ poundals *towards the centre.*

The particle has a constant acceleration $\frac{v^2}{r}$ celos towards the centre.

Acceleration never exists unless force causes it; hence there is a force $\frac{mv^2}{r}$ poundals towards the centre.

270. A force which always tends towards a fixed point is called a **centripetal** force.

Example. *A particle of* 1 lb., *fastened to a fixed point* O *by a light string* 2 ft. long, *describes the circle about* O *as centre once every second; find the* tension *of the string.*

[NOTE. In a question like this, gravity is not supposed to be acting. The particle might be moving on a perfectly smooth horizontal plane. The student should notice that unless some word such as, *weight, horizontal, vertical, heavy,* appears in a question, gravity is supposed not to act.]

The particle describes the circumference of a circle of 2 ft. radius in 1 sec.; therefore it moves at the rate of 4π ft. per sec.

Therefore by Art. 142 its acceleration is $\frac{16\pi^2}{2}$ celos.

Therefore it is acted on by $8\pi^2$ poundals.

The tension of the string is therefore equal to the weight of about

$$\frac{8 \times 22 \times 22}{32 \times 49} \text{ pounds, nearly;}$$

that is, to the weight of 2·47 lbs. nearly.

NOTE. When a mass is describing a circle with uniform speed, its velocity is being changed (not in magnitude, but) in direction. By reason of its inertia this change cannot be made in the velocity of the mass by anything except by external force. We have shewn that in order to effect this change a force of a certain magnitude must continuously act on the particle *towards the centre* of the circle. The inertia of the mass continuously opposes this force; and used to be said to be *centrifugal.* This word *centrifugal* is very misleading; for the student should carefully notice that if the *centripetal* force ceases, the motion of the mass is along a *tangent*: therefore the tendency of the mass should be called **tangential**, not centrifugal.

CIRCULAR MOTION TREATED BY THE HODOGRAPH.

271. A diagram of displacement is the representation by a figure, of the path of a moving point for some interval.

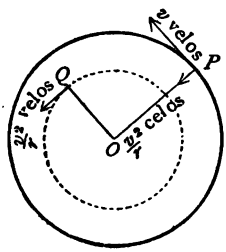
A diagram of velocity is the representation by a figure of the velocity of the moving point for some interval.

272. DEF. Let OQ represent the velocity of the point P at any instant. If we consider the various positions of Q for a certain interval then Q will trace out a curve. This curve is called the **Hodograph** of the point P.

This curve together with the point O is a diagram of velocity.

273. PROP. *The* hodograph *of a point moving with constant velocity in a circle is a* circle.

Let O be the centre of the circle described by the point P.



Draw OQ perpendicular to OP and take the length of OQ to represent the velocity of P.

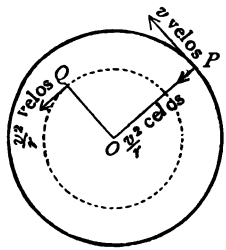
Now as P describes its circle OQ which is always parallel to the velocity of Q is always perpendicular to OP and is of constant length (for the velocity of P is constant). Hence the path of Q is a circle, and Q describes this circle with uniform velocity.

274. PROP. *The acceleration of the point* P *is numerically equal to the velocity of* Q *in the hodograph.*

For the displacement of Q in a small interval represents the additional velocity of P for that interval.

The acceleration of P is the time-rate of its additional velocity.

Therefore the velocity of Q (which is its time-rate of displacement) is the time-rate of the additional velocity of P, that is, is the acceleration of P.



275. *Example.* When P describes a circle of radius r feet uniformly with velocity v velos, its acceleration is $\frac{v^2}{r}$ celos, towards the centre of the circle.

From the last example the velocity of Q is uniform and is parallel to PO.

Therefore the acceleration of P is constant in magnitude and always in the direction from P to O.

Q describes its circle while P describes its circle,

$$\therefore \text{ velocity of } Q : \text{velocity of } P = OQ : OP,$$

that is,

$$\frac{\text{number of celos in accel. of } P}{\text{number of velos in vel. of } P} = \frac{\text{no. of velos in vel. of } P}{\text{no. of feet in } OP},$$

$$\therefore \text{ acceleration of } P = \frac{v^2}{r} \text{ celos,}$$

and it is in the direction PO.

276. When a particle of mass m pounds describes a circle of radius r feet with constant velocity v velos, it has the acceleration $\frac{v^2}{r}$ celos towards the centre of the circle.

It is therefore being acted on by the force $\frac{v^2}{r} \times m$ poundals towards the centre of the circle.

When a particle is whirled round in a circle on a smooth horizontal plane it must be retained in the circular path by some force. This force can be produced by the tension of a string fastened to a nail fixed at the centre of the circle.

If the plane were quite smooth the velocity of the particle would be constant and the tension of the string constant.

The tension of the string is a stress, and one part of the stress acts on the nail pulling it outwards, the other part acts on the particle pulling it inwards. If the string at any instant be cut, the particle would cease to be under the action of force and would move in a straight line.

Example. An engine of 60 tons *is going round a circular railway curve of* 200 yards *radius at the rate* 30 miles an hour, *find the horizontal pressure of the rails on the wheels.*

Here the horizontal force $= \frac{mv^2}{r}$ poundals; $m = 60 \times 2240$ pounds; $v = 44$ celos; $r = 600$ feet.

$$\therefore \text{the force} = \frac{60 \times 2240 \times 44 \times 44}{600} \text{ poundals}$$

$$= \frac{224 \times 44 \times 44}{32} \text{ pounds weight}$$

$$= \text{the weight of } 121 \text{ cwt.; or, of } 6 \text{ tons } 1 \text{ cwt.}$$

EXAMPLES. XLIX.

In the first four of the following questions gravity is not acting.

1. A point is describing a circle of radius 5 ft. with uniform speed 10 velos, find its acceleration.

2. A mass of 4 lbs. is describing a circle of radius 5 ft. with uniform speed of 10 velos; what force is acting upon it?

3. A mass m lbs. fastened by a string to a fixed point is describing a circle uniformly with v velos; the tension of the string is p poundals, find the length of the string.

4. A mass m lbs. is fastened by a string of length r feet to a fixed point O, and describes a circle about O n times per second; shew that the tension of the string is $4mn^2\pi^2r$ poundals.

5. A railway carriage is describing a curve on horizontal rails; shew that if it goes fast enough it must fall over.

6. A railway carriage of 10 tons is describing an arc of a circle of $\frac{1}{4}$ mile radius with a speed of 30 miles an hour; find the horizontal force which must be acting on the carriage.

7. A mass of 1 cwt. is in a railway truck which is moving, on horizontal rails on a curve whose radius is 400 yards, at the rate of 60 miles an hour; what is the horizontal stress between 1 cwt. and the truck?

8. A mass of 56 lbs. is placed in a swing the ropes of which are 10 ft. long; find the difference in the tension of the ropes when the swing is at its lowest point, (i) when it is at rest, (ii) when it is moving with 20 velos (neglecting the mass of the swing itself).

9. A horse weighing 5 cwt. is going along a road at the rate of 10 miles an hour and passes over a depression in the road whose section is a circle of 40 feet radius; what is the difference between his weight and the average pressure of his feet on the road when at the lowest point of the depression?

10. Two masses of 3 lbs. and 4 lbs. on a smooth horizontal plane are fastened together by a light inelastic string 7 ft. long and are each describing a circle so that one point in the string is at rest on the plane; which must be the point in the string?

11. Two masses A and B of 1 lb. each are joined by a string; the mass A describes a circle of radius 2 ft. with uniform speed on the surface of a smooth horizontal table, while the other mass B is suspended under the action of gravity by the string, which passes through a small hole in the table at the centre of the circle; find the speed of A.

12. Two masses A and B are joined by a string; the mass A of 10 lbs. describes on a smooth horizontal table a circle of radius 5 ft. with uniform speed 5 velos; the other mass B is suspended under the action of gravity by the string which passes through a small hole in the table at the centre of A's circle; find the mass of B that it may rest in equilibrium.

13. A particle of 3 lbs. is fastened to a fixed point by a string $2\sqrt{2}$ feet long and is describing a horizontal circle under the action of gravity, so that the string describes a right circular cone, the inclination of the string to the vertical being 45^0; prove that the velocity of the particle is about 8 velos.

14. Shew that the tension of the string in Question **13** is $3\sqrt{2}$ lbs. weight.

SECTION VII.

ENERGY.

CHAPTER XXI.

WORK.

277. An external force is said to be doing **work** on a mass when its point of application is undergoing displacement in the direction of the force.

Examples. The weight of a mass falling freely does work on the mass.

The pressure of the foot of a bicyclist does work on the treadle of his bicycle.

The pull of a horse does work on the carriage it is drawing.

When a horse is attached to a railway truck and pulls at it, then as long as the truck is *at rest*, we say the horse is *leaning* against the truck, and does *no work;* but when the truck *moves* and the horse continues his pull, he does work as long as his pull is continued.

278. DEF. The **work** done in an interval on a mass by an external force, varies directly as the force; and also directly as the resolved part in the direction of the force, of the displacement for that interval of the point of application of the force.

Thus, the work done by a force applied to a mass in motion, is doubled, trebled, etc., when the force is doubled, trebled, etc., for the same displacement; and the work done by a given force is doubled, trebled, etc., when the resolute of its displacement in the direction of the force is doubled, trebled, etc.

It is convenient to refer to the resolute in the direction of the force, of the displacement of the point of application of a force, as *the displacement through which the force works.*

NOTE. The work done by an external force on a moving mass is *independent of the length of the interval occupied;* it depends only on the *force* and the *displacement* of the mass.

279. DEF. We shall choose as our **unit** *work* the *work* done by a *poundal* while its point of application is displaced a *foot* in the direction of the poundal.

We shall call this *unit* work a **foot-poundal**.

Hence the **measure** of the work done by a uniform force, is the product of the *number* of poundals by the *number* of feet through which it works.

Example. The work done by the *weight* of 1 lb. while the 1 lb. is displaced 1 foot vertically downwards, is g foot-poundals.

For, the weight of 1 lb. $= g$ poundals,

1 poundal acting through 1 ft. $= 1$ foot-poundal,

g poundals acting through 1 ft. $= g$ foot-poundals.

280. When the displacement through which the force works, is in the direction *opposite* to that of the force, the displacement is *negative.* In this case the work done by the force is said to be **negative**; or we say that work is done *against* the force.

When a mass under the action of gravity is raised vertically, work is done *against* the weight of the mass.

281. The work done against gravity in raising 1 lb. through the vertical displacement 1 foot, is g foot-poundals.

This amount of work is called a **foot-pound**, which is an abbreviation for a **foot-pound weight**.

Hence *one foot-pound* is equal to g *foot-poundals.*

Example. Find the work done *by the weight of* 1 cwt. *while the* cwt. *falls a vertical distance of* 10 feet.

The force is 32×112 poundals.

The displacement of the mass in the direction of the force, that is, the displacement through which the weight works, is 10 ft.

The work done is therefore $32 \times 112 \times 10$ foot-poundals.

282. When two equal and opposite forces (as, for example, the action and reaction of a stress) act upon the *same* point the work done by one of the forces is equal and opposite to the work done against the other.

Also, if two equal and opposite forces act upon different points, which points have no motion *relative to each other*, the work done by one of the forces is equal and opposite to that done against the other force.

The total work done by two such forces is zero.

Example i. The tension of an inextensible string does no work.

A man in a railway carriage who presses with his feet against the opposite seat does no work, notwithstanding that the mass, to which he applies two equal and opposite forces, is in motion.

Example ii. When the masses in an Atwood's machine are *equal* and are in motion the weight of one does as much work on its mass as the second mass does against its own weight.

Example iii. 1 cwt. *slides down an inclined plane of height* 10 ft. *and inclination* α; *find the* work done *by the* weight *of the mass when the mass goes from the top to the bottom.*

The distance passed over by the mass is the slant side of the plane; the direction of the force is vertically downward; the resolute of the displacement in the direction of the force is the vertical displacement; this is, the height of the plane, viz. 10 feet.

Hence, the work done is $10 \times 112 \times 32$ foot-poundals;

that is, 1120 foot-pounds.

Example iv. *A horse draws a tram-car of* 2 tons *along a horizontal road with the uniform velocity of* $7\frac{1}{2}$ miles an hour; *supposing the resistance of the friction etc. is* $\frac{1}{56}$ *of the weight of the car, find how many* foot-pounds *the horse is doing* per second.

The resistance of friction etc. is a force equal to the weight of $\frac{1}{56}$ of 2 tons; that is, the weight of 56 lbs.

Therefore the horse must be applying a force of $56 \times g$ poundals; [for, since the velocity of the car is uniform, no additional force is required by the inertia of the car.]

The point of application of this force is moving at the rate of $7\frac{1}{2}$ miles per hour; that is, of 11 velos.

Therefore the horse is doing $11 \times 56 \times g$ foot-poundals per second;

that is, 616 foot-pounds per second.

283. PROP. *The number of* foot-pounds-wt. of work done *by the* weight of *a mass when the mass is displaced, is the product of the* number of lbs. weight *by the* number of feet *in the* **vertical** *displacement* of the mass.

Since the force which is working is always vertical and downwards, the displacement through which it works is always the vertical resolute of the actual displacement of the mass. The work done by the weight is positive when the vertical displacement is downwards and negative when it is upwards.

284. PROP. *The measure of the work done by the weights of any given particles, when they are displaced, is the* **product of** *the number of lbs.-wt. in* **the sum of their weights by** *the number of feet in* **the vertical displacement of their Centre of Gravity.**

For with the usual notation [see Chap. XII., Art. 162], if y_1, y_2, y_3,... be the vertical displacements of the particles m_1, m_2, m_3,... and $\bar{y}$ that of their Centre of Gravity, then the work done is $y_1m_1g + y_2m_2g + y_3m_3g + \ldots$

and this is equal to $\bar{y}\,(m_1 + m_2 + m_3 + \ldots)$

which is the work done supposing the whole mass is concentrated at its Centre of Gravity. Q.E.D.

Thus, the work done in raising 1 ton through a vertical distance 50 feet is the same, viz. $2240 \times 50 \times g$ foot-poundals, by whatever path the ton arrives at that height. It may be raised by screws; or by a system of pulleys; or by a horse pulling it up an incline in a cart; or it may be raised by a man carrying it piece-meal up a ladder.

EXAMPLES. L.

Find the number of foot-poundals and of foot-pounds done by the weights of the following.

1. 56 lbs. in descending 40 feet.
2. A man of 10 stone in descending 500 feet.
3. A block of stone of 10 tons in descending 500 feet.
4. A man of 12 stone in ascending a tower 200 ft. high.
5. A train of 300 tons in descending an incline of 1 in 50, one mile in length.
6. A waggon and horses of 3 tons in descending a hill, whose inclination is 1 in 30, for a distance of 200 yards.

7. Compare the work done in 1 hour against gravity (i) by a man who lifts 28 lbs. a height of 3 ft. every minute, (ii) by a man of 10 stone who walks up an incline of 1 in 6 at the rate of 4 miles an hour.

8. Find the work done against friction in moving a ton 100 feet along a rough horizontal plane the horizontal resistance of the friction being $\frac{1}{10}$ of the weight.

9. Find the number of foot-pounds per second required to draw a cart of 1 ton along a horizontal plane uniformly at the rate of 5 miles an hour, the resistance due to friction etc. being $\frac{1}{10}$ of the weight.

10. Find the greatest uniform velocity with which an engine capable of producing 140,000 foot-pounds of work per second, can draw a train of 100 tons along a horizontal plane, the resistance due to friction etc. being $\frac{1}{80}$ of the weight.

11. An engine is drawing a train of 120 tons up a smooth inclined plane of 1 in 60 at the rate of 24 miles an hour; how much work is it doing per second (friction to be neglected)?

12. Supposing the resistance due to friction etc. in Question **11** to be equivalent to the weight of 1120 lbs. and to act opposite to the direction of motion, find at what rate the engine could draw the same train up an incline of 1 in 60, doing the same amount of work per second as before.

13. Supposing the resistance due to friction in Question **12** to be $\frac{1}{10}$th of the *pressure* of the train on the inclined plane, find at what rate the engine could draw the train up the inclined plane with the same expenditure of work per second.

14. Supposing that a man of 12 stone in walking raises his whole weight a vertical distance of 1 inch every step, and that the length of each step is $2\frac{1}{2}$ feet, find how much work the man does in this way in walking 1 mile.

15. Compare the work done by the man in Question **14** with that done by another man of 12 stone going a mile with uniform speed on a bicycle weighing $\frac{1}{2}$ cwt., the resistance due to friction etc. of the bicycle being $\frac{1}{100}$ of the weight.

CHAPTER XXII.

ENERGY.

285. DEF. **Energy** is *capacity for doing work.*

There are two different forms of energy, viz. *Potential Energy* and *Kinetic Energy.*

286. DEF. **Potential** *energy* is the capacity for doing work which a mass has by virtue of its *position.*

A spring when *bent* has energy by reason of its position; for it can do work when changing that position.

Air when compressed can do work on being allowed to expand.

A mass at the top of a tower can do work in its descent to a lower level.

Example. Find the difference between the Potential Energy *of* 1 cwt. *at the* top *of a tower* 100 ft. *high, and that of* 1 cwt. *at the foot of the tower.*

In descending from the top of the tower to the bottom, the *weight* of the 1 cwt. does 100×112 foot-pounds;

Hence, 1 cwt. at the top of a tower 100 ft. high has 100×112 foot-pounds more Potential Energy that 1 cwt. at the foot of the tower.

287. DEF. **Kinetic** *energy* is the capacity for doing work which a mass has by virtue of its mass-velocity.

A bullet when in motion can do work in giving up its velocity.

Water when in motion can do work in turning a water wheel.

The mass of a hammer can do work by virtue of its velocity.

The kinetic energy of a particle having a given velocity is measured by the number of foot-poundals it does in giving up that velocity.

288. PROP. *Prove that the difference between the energy of* m lbs. *having* v velos, *and that of* m lbs. *having* u velos, *is* $(\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$ foot-poundals.

Let a force of p poundals be applied to the m lbs. so that in t seconds its velocity is reduced from v velos to u velos; also suppose that in that interval the mass has a retardation α celos and is displaced s feet in the direction opposite to the force; then, by Art. 25. iii,

$$\alpha s = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2, \text{ or, } mas = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2;$$

but $m\alpha$ is the number of poundals required to produce in m lbs. acceleration α celos; so that $m\alpha = p$.

Therefore $$ps = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$

But ps is the number of foot-poundals done by p poundals which acts on a mass while it passes over s feet in its own direction. That is to say, the mass m lbs. having v velos can do $(\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$ foot-poundals while its velocity is reduced from v velos to u velos. Q.E.D.

289. The above proposition is true when the force acting on m is not uniform, for if a series of uniform forces $p_1, p_2, p_3 \ldots p_n$ work through displacements $s_1, s_2, s_3, \ldots s_n$, on a particle m we have

$$\begin{aligned} &s_1p_1 + s_2p_2 + s_3p_3 + \ldots p_ns_n \\ &\quad = (\tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mu^2) + (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2) + (\tfrac{1}{2}mv_3^2 - \tfrac{1}{2}mv_3^2) + \ldots \\ &\quad = \tfrac{1}{2}mv_n^2 - \tfrac{1}{2}mu^2. \end{aligned}$$ Hence

290. The kinetic energy of a particle of m lbs. which has v velos is $\frac{1}{2}mv^2$ foot-poundals.

For the mass can do $\frac{1}{2}mv^2$ foot-poundals against a force before it is reduced to rest.

Also, if $\frac{1}{2}mv^2$ foot-poundals of work are done on a particle of mass m lbs. at rest, it will thereupon have v velos.

Example. Find the work done by the engine of a train of 120 tons, *moving the train a* mile *from rest and at the same time giving it a speed of* 30 miles per hour; *the motion being on a horizontal line and the resistance due to friction* $\frac{1}{12}$ of the weight of the train.

The work done against friction is

$$(\tfrac{1}{12} \times 2240 \times 120) \times (1760 \times 3) \text{ foot-pounds}$$
$$= 1120 \times 1760 \times 3 \text{ foot-pounds.}$$

The work done on the inertia, in producing kinetic energy,

is $\frac{1}{2} \times 120 \times 2240 \times (44)^2$ foot-poundals [Art. 282.]

$$= \tfrac{1}{32} \times \tfrac{1}{2} \times 120 \times 2240 \times (44)^2 \text{ foot-pounds.}$$

Hence, the whole work done by the engine is

$$(1120 \times 30 \times 176 + 1120 \times 30 \times 2 \times 121) \text{ foot-pounds}$$
$$= 1120 \times 30 \times 418 \text{ foot-pounds.}$$

EXAMPLES. LI.

Find the kinetic energy of each of the six following:

1. A stone of 4 lbs. which has 200 velos.
2. A cannon shot of 8 cwt. which has 2000 velos.
3. A cannon shot of 12 cwt. which has 1600 velos.
4. A train of 400 tons going at the rate of 60 miles an hour.
5. A man of 12 stone running at the rate of 12 miles an hour.
6. A man of 12 stone after falling a distance of 10 feet.
7. Find the work done by the engine of a train of 360 tons, in going 2 miles from rest on horizontal rails, the friction being $\frac{1}{12}$ of the weight, and the speed attained 45 miles per hour.
8. Find the work necessary to raise a train of 100 tons to the top of a mountain pass 500 feet high and at the same time to give it a velocity of 60 miles an hour (neglecting friction etc.).
9. A man has to raise 1 cwt. of bricks 8 feet; he throws them up so that they arrive at a point 8 ft. high with 5 velos; compare the superfluous work done with the necessary work.
10. A tram car of 1 ton is stopped by a brake 10 times in going a mile; the brake stops the car in 11 yards; after each stoppage the car attains a velocity of $7\frac{1}{2}$ miles an hour. Supposing the friction of the rails to be a uniform force of 28 lbs. weight, compare the work done in this journey with the work done in going a mile with uniform velocity. [At the beginning of the mile the car is moving at the rate of $7\frac{1}{2}$ miles an hour.]

11. The expense of moving a train is proportional to the work done. Compare the cost of getting the speed of a train up from rest to a velocity of 45 miles an hour and at the same time going 1 mile, with the cost of moving it a mile with uniform velocity; supposing that the resistance caused by friction etc. to be $\frac{1}{170}$ of the weight of the train.

12. Suppose the tram-car in Question 10 had no brake, and so had to be stopped by the horses; find how much more work the horses would have to do in consequence in a journey which includes 10 stoppages.

291. When a particle slides along a *smooth* **plane** the pressure of the plane, whatever it is, is always perpendicular to the plane, and therefore perpendicular to the path of the particle; the *displacement* through which the pressure *works* is therefore zero; hence

292. The pressure of a smooth plane on a particle moving along its surface does no work on the particle.

Example. When a particle slides down a smooth inclined plane under the action of gravity only, its total energy *is unchanged.*

By Note Art. 239, a particle of m lbs. sliding down an inclined plane of angle α has $g \sin \alpha$ celos.

Let h feet be the *vertical* displacement of the particle in any interval t seconds during which the velocity of the particle is changed from u velos to v velos.

Then the particle has lost mgh foot-poundals of potential energy and gained $(\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$ foot-poundals of kinetic energy; we have to prove that these amounts of work are equal.

Let s feet be the displacement on the plane,

then $$s(g \sin \alpha) = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2,$$

but $$h = s \sin \alpha.$$

Therefore $$hg = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2.$$ Q. E. D.

293. The sum of the potential energy and of the kinetic energy of a particle is unaltered by any motion under the action of gravity only.

Example i. Let m lbs. fall from rest h feet *vertically*. Then the potential energy is diminished by mh foot-pounds.

It receives v velos, where $mgh = \frac{1}{2}mv^2$; so that it has received kinetic energy capable of doing mgh foot-poundals, that is, mh foot-pounds; in other words, the potential energy lost is equal to the kinetic energy gained.

Example ii. Since the velocity of a projectile is equal to that due to a fall from the directrix, it follows that (the kinetic energy + the potential energy) of a projectile is always equal to the potential energy of an equal particle at rest at some point on the directrix.

294. When a particle slides along a **curved surface** which is *smooth*, it is understood that the pressure of the surface on the particle is always perpendicular to the surface, and therefore this pressure is *always* perpendicular to the path of the particle.

The *displacement* through which the pressure works is therefore *always* zero; so that the *work* done by the pressure is zero; hence,

295. *When a particle is sliding on a smooth surface under the action of no forces, its kinetic energy is constant;* consequently *its speed is constant.*

296. Since the above proposition is true of a circle of *any* radius, and of an arc of any magnitude, it must be true for *any* continuous curve.

For such a curve may be considered to be made up of successive small arcs of its successive circles of curvature.

297. When a particle slides on a smooth continuously curved surface, under the action of gravity only, its total energy is constant.

For, the only forces acting are the pressure of the surface and the weight of the particle; and the pressure of the surface does not alter the kinetic energy of the particle, while the weight of the particle produces its own effect, no matter what other forces may be acting. The effect of the weight is expressed by saying that the total energy of the particle is constant.

Example i. *A smooth fine wire, in the form of a circle of radius* r ft. *is placed in a vertical position, and a particle in the form of a small ring slides on it under the action of gravity. The ring has* u velos *when at the highest point of the circle; find its* velocity at the lowest point.

Let the ring contain m lbs.

In going from the highest point to the lowest it loses

$2rmg$ foot-poundals of potential energy.

Let its velocity at the lowest point be v velos; it will therefore have gained $(\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$ foot-poundals of kinetic energy.

But on the whole, its energy is unchanged.

Therefore, $2rmg = (\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$,

or, $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + 2rmg$.

Example ii. *A stone is fastened by a light inextensible string* 1 foot *long, to a fixed point, and is describing a vertical circle about the fixed point as centre, under the action of gravity; find the* least possible velocity at the lowest point of the circle.

The constraint of the string is similar to that of a smooth circular wire; for the tension of the string, being always perpendicular to the direction of motion, does no work; hence, by Art. 286, the total energy of the stone is constant.

Therefore, if v velos be the velocity of the stone at the lowest point of the circle, u velos its velocity at a point on the circle whose vertical distance above the lowest point is h feet, then

$$mgh = (\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2),$$

or,
$$\tfrac{1}{2}v^2 - \tfrac{1}{2}u^2 = gh \quad \text{.....(i).}$$

Now, in order that the stone may completely describe the circle, the tension of the string (which is *least* at the highest point) must just vanish at that highest point.

Let u velos be the velocity of the particle at the highest point, then its acceleration is $\frac{u^2}{r}$ celos vertically downwards.

The only force acting when the stone is at the highest point of the circle, is the weight, mg poundals vertically downwards; for the only other force available is the tension of the string, which at this moment is zero.

Hence,
$$mg = \frac{mu^2}{r} = mu^2 \text{ [for } r = 1\text{]} \quad \text{.....(ii).}$$

Therefore, since v is the velocity at the lowest point of the circle, putting $h = 2$ ft. in (i) and combining with (ii), we have

$$v^2 = u^2 + 4g = g + 4g = 5g.$$

Hence, the least velocity at the highest point is about $4\sqrt{2}$ velos, and the corresponding velocity at the lowest point about $4\sqrt{10}$ velos.

298. The unit of work in the **C.G.S.** system of units [see Art. 78] is the work done by a Dyne working through a Centimetre.

This unit work is called an **Erg.**

EXAMPLES. LII.

1. A particle is describing a parabola under the action of gravity, prove that its total energy is constant.

2. A heavy particle slides down a smooth straight tube from rest and then describes a parabola; find its velocity when it is 100 feet below the point from which it started.

3. A perfectly elastic particle is let fall on a horizontal pavement, and rebounds; prove that the total energy of the particle is constant.

4. Two perfectly elastic spheres impinge directly, prove that their total energy before and after impact is the same.

5. Two perfectly elastic spheres impinge obliquely, prove that their total energy is unaltered by the impact.

6. An imperfectly elastic particle falls on a horizontal plane and rebounds; shew that it loses energy by the impact.

NOTE. A **simple pendulum** consists of a heavy particle fastened to a fixed point by a light rod; the particle moves in the lowest arc of a vertical circle under the action of gravity.

7. A simple pendulum *of length* l feet swings through an arc 4α, prove that its velocity at the lowest point of its path is $2 \sin \alpha \sqrt{(gl)}$ velos.

8. A circular wheel, whose rim consists of a heavy uniform wire of m lbs. and whose spokes, (of length r feet), and axle are such that their mass may be neglected, revolves about its axis, which is fixed, so as to make n complete revolutions per second; shew that the kinetic energy of the wheel is $2mn^2\pi^2r^2$ foot-poundals.

9. A fine light string is coiled round the wheel of Question **8**, one end being fastened to the rim and the other end supporting a heavy mass m' lbs. This mass descends from rest under the action of gravity, causing the wheel to turn as it descends; prove that when m' has descended h feet from rest its velocity is

$$\sqrt{\frac{2m'gh}{m+m'}} \text{ velos.}$$

CHAPTER XXIII.

POWER.

299. We use the word '**agent**' to denote a *machine* which transforms some stored-up chemical or other energy into work.

Examples. A *navvy* transforms some of the chemical energy of his food into work when he raises an embankment.

A *steam-engine* transforms some of the energy stored in coal into work when it does work on a train.

A *windmill* transforms some of the kinetic energy of the air into work when it pumps up water, etc.

300. DEF. The **power** of an agent varies as the amount of work which the agent can produce from the energy supplied to it in a unit of time.

The **unit** *power* is the power of an agent which can transform 1 foot-poundal per second.

Hence the **measure** of the power of an agent is the *number* of foot-poundals which it can transform per second.

301. A **Horse-power** is the *power* of an agent which can transform 550 foot-pounds per second;

that is, $550 \times g$ foot-poundals per second.

Example 1. *At what uniform speed can a* horse *of* 1 horse-power *draw a tram-car of* 1 ton, *supposing the friction etc. to cause a uniform resistance of* 50 lbs. weight?

Suppose the required speed to be v velos.

Then the tram passes over v feet per second.

Therefore the horse must produce $v \times 50$ foot pounds per second.

He can produce 550 foot pounds per second.

Hence $v = 11$.

That is, he can draw the car at the rate of 11 velos;

or, at the rate of $7\frac{1}{2}$ miles per hour.

Example ii. *At what uniform* speed *can a man of* $\frac{1}{10}$ horse-power *draw* 1 ton *up a smooth inclined plane of* 1 in 10?

The sine of the angle of inclination is $\frac{1}{10}$.

Therefore the force necessary to move the ton up the plane is $\frac{1}{10}$ of a ton weight; or, 224 lbs. weight.

[Now it is very unlikely that the man can exert such a force, unaided.

To do so, he must make use of a wheel and axle, or some such mechanical contrivance, which by multiplying the distance through which the muscular force of the man works, divides by the same number the force which must be exerted.]

Let the velocity required be v velos,

then, $v \times 224 = 55$,

or, $v = \frac{55}{224} = \frac{1}{4}$ (very nearly).

That is, he can (by the aid of a machine) draw the ton up the plane at the rate of about 15 feet per minute.

302. The following is an illustration of the meaning of the terms work, energy, power.

Suppose a Contractor undertakes to excavate a certain amount of earth to form a canal or dock. The raising of the earth and the moving it from one place to another requires the doing of a certain amount of work against gravity and against friction. He will bring together a certain number of labourers and steam-engines, and these together are the *power* he proposes to use. But no matter how much power he brings together he can do none of the work (that is, he can excavate no earth) until he has brought up energy in the form of food and fuel. He may use as much power as he pleases and so get the work done quickly or slowly according as the amount of the power he provides is greater or less. But he must provide energy (stored either in food or in fuel) in proportion to the amount of the work he has undertaken to do.

303. It must not be supposed however that a steam-engine produces in the form of useful work more than a small proportion of the energy stored in the coal or other fuel with which it is fed.

A man is a much more **efficient** engine than a steam-engine; that is, a man can produce in the form of useful work a much larger proportion of the energy stored in his food than the steam-engine can of the energy stored in its fuel. But a steam-engine is a cheaper worker than a man because energy in the form of coal is much cheaper than energy in the form of food.

304. DEF. The number of foot-pounds of work yielded by a steam-engine in consequence of the consumption of 1 bushel (i.e. 84 lbs.) of coal is called the **duty** of the engine.

The average duty of a good low-pressure steam-engine is about 90 millions.

EXAMPLES. LIII.

1. What must be the power of an engine that can keep a train of 100 tons going at the rate of 60 miles an hour on horizontal rails when the friction is equivalent to a horizontal resistance of $\frac{1}{50}$ of the weight of the train?

2. At what uniform rate could the engine of Question 1 draw the same train up an incline of 1 in 50, the resistance of friction etc. being unaltered?

3. In how many minutes would an engine of 5 horse-power raise 12 cwt. a vertical height 192 feet?

4. What uniform velocity can an engine of 60 horse-power maintain in a train of 90 tons against resistances of friction etc. of 8 lbs. (weight) per ton?

5. A horse can draw a cart of 1 ton at the rate of 4 miles an hour; supposing that his power is 1 horse-power, find the resistance of friction.

6. What is the weight of a train in which the engine of 50 horse-power can maintain a uniform velocity of 25 miles per hour against resistances of 8 lbs. wt. per ton?

7. What must be the power of an engine which can raise 1000 gallons of water per minute a height of 20 feet, delivering it by means of a uniform pipe of which a linear foot contains 1 gallon? [*A pint of water weighs a pound and a quarter.*]

8. A man throws a brick of 7 lbs. a height of 8 feet once in a second; at what rate is he working?

9. The length of the stroke of the piston of an engine is 7 ft. 4 in.; the mean pressure of the steam is 12 lbs. (wt.) per sq. inch; the number of strokes made by the piston is 20 per minute; what must be the surface of the piston in order that the horse-power of the engine may be 80?

10. The internal diameter of the cylinder of a steam-engine is 1 ft. 6 in.; the length of the stroke is 4 ft.; it makes 18 strokes per minute; under what average pressure of steam per square inch must it work in order that work at the rate of 75 horse-power may be done on the piston?

11. It is found that a certain water-wheel gives out in useful work ·55 of the potential energy lost by the water in passing over it. If the average section of the stream is 5 ft. by 2 ft.; its velocity 1 velo; and a fall at the wheel of 12 ft.; what is the horse-power of the machine?

12. How many bushels of coal must be expended per hour in raising 600 cubic feet of water per minute a height of 850 feet, the duty of the engine being 75 millions?

CHAPTER XXIV.

ENERGY IS INDESTRUCTIBLE.

305. When a machine does work it receives a supply of some kind of energy which it transforms into mechanical work.

A man who raises weights receives a supply of energy by the chemical combustion of the food which he eats.

306. A machine which has no energy supplied to it cannot do work; it can only *transmit* work.

A system of pulleys used in raising weights transmits the work given to it by what is termed, by many writers on Statics, '*the power*'; this work is transmitted to what is called '*the weight*.'

It will be found that (when the kinetic energy is unchanged and friction is neglected) the work done by '*the power*' is always equal to the work done on '*the weight*.'

For example, if the weight is 16 times the power, the power will through the displacement work 16 feet in raising the weight 1 foot.

307. Since in a *frictionless* machine, whose parts are moving with constant speed, the amount of work put into the machine is equal to the amount of work which the machine gives out, we can often find the ratio of the *power* to the *weight* by the following method.

Suppose it can be ascertained through what distance (d) the '*power*' works in a given short interval of time, and also through what distance (d') work is done on the weight in the same interval; then if the power be p poundals and the weight Q poundals, the work done by the power is $P \cdot d$ *foot-poundals*; the work done on the weight is $Q \cdot d'$ *foot-poundals*; therefore $p \cdot d = Q \cdot d'$; whence, $\frac{p}{Q} = \frac{d'}{d}$.

308. When the machine is such that the ratio of the Power to the Weight is constant, this ratio will be independent of the interval observed. When the ratio of the Power and the Weight varies (*the kinetic energy of the machine being constant*) then the result obtained above will be more and more nearly true at a given instant, the smaller the interval during which the distances d, d' are observed.

Hence the ratio of the power to the weight is the limiting value of the ratio of the distances d, d' when the interval of observation is infinitely diminished.

309. When the kinetic energy is 'virtually' zero the method becomes a Statical Principle of great importance, and is called **the Principle of Virtual Work.**

310. No machine however can be made in which the resistance of *friction* is entirely excluded; so that no machine can transmit the *whole* of the work put into it.

This friction, which absorbs work, uses the work it absorbs in heating the parts of the machine; it changes some of the work put into the machine into another form of energy which we call *heat.*

311. Energy is indestructible; therefore energy cannot be produced out of nothing; for that which can be produced can be destroyed. One form of energy can be obtained from another; but it can be obtained in no other way.

A steam-engine does work; it is supplied with the energy of heat obtained from the combustion of coal; in which coal is stored the energy radiated from the sun countless years ago.

312. Mechanical work involves the moving of the surface of one mass over that of another. This motion invariably sets up friction which absorbs energy, converting it into heat. Heat tends to diffuse itself in all directions. And since heat can only be converted into mechanical energy when the heat is highly *concentrated*, it follows that the energy absorbed by friction is dissipated in such a way that it is lost for all practical purposes.

313. Hence, notwithstanding that energy is indestructible, yet kinetic energy is continually being eaten away, as it were, by friction and converted into the unserviceable form of uniformly diffused heat.

For example, it is proved in the *Dynamics*, Art. 133, that the kinetic energy of two masses is diminished by their mutual impact. The reason of this loss of kinetic energy is that during an impact the particles of the masses in the neighbourhood of the point of impact are first compressed and then recover more or less their original position. This compression and recovery generates *heat;* accordingly the kinetic energy of the two masses is diminished by the amount of energy taken up by the heat generated.

314. Heat is itself the kinetic energy of the minute particles of a substance.

Gas consists of minute invisible particles of matter; these particles are in motion; they impinge repeatedly and ceaselessly on one another. They are perfectly elastic (possibly because they are too small to be capable of compression); consequently by Art. 133 of the *Dynamics* the joint energy of two particles is unaltered by their impact.

The quantity of heat contained in a certain amount of gas is *the kinetic energy* of all its particles. This energy (if none be allowed to pass away from the gas) continues unaltered because the particles of the gas are perfectly elastic.

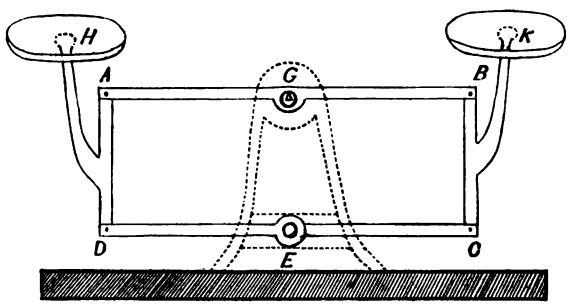
315. The main principle which it is the object of Dynamics to explain and establish may be summed up in the words **Energy is indestructible.**

'The discussion of the various forms of energy—gravi-'tational, electro-magnetic, molecular, thermal &c.—with 'the conditions of the transference of energy from one form 'to another, and the constant dissipation of the energy 'available for producing work, constitutes the whole of 'physical science,—in so far as it has been developed in 'the dynamical form,—under the various designations of 'Astronomy, Electricity, Magnetism, Optics, Theory of the 'Physical states of bodies, Thermo-dynamics and Chemistry.' *Professor Maxwell's Matter and Motion*, p. 95.

316. The principle of Virtual Work affords a simple proof of the principle of

ROBERVAL'S BALANCE.

This machine consists of a symmetrical bar AB, resting on a knife-edge at its middle point G; and four other bars BC, CE, ED, DA smoothly hinged together as in the figure; E is an axis fixed vertically below the knife-edge G. AD and BC are exactly equal, as also are DE and EC. Also $BCDA$ is a parallelogram; hence BC and AD are both parallel to GE and are therefore each of them *vertical*, for all positions of the bar AB.



Two scale-pans of equal weight are affixed, one to each of the rods AD, BC.

Then, when the bar AB receives any displacement whatever, the work done by reason of the displacement on the side of the machine which is raised, is exactly equal to the work done by the other which is lowered.

The scale-pans need not be symmetrical in shape, for the *vertical* motion of each particle of each pan is the same.

Now, if weights are placed one on each scale-pan, the necessary condition for the equilibrium of the machine is, that when a displacement is made the resultant work done on the two weights should be zero. This will be the case provided the weights are equal, for the vertical displacements of the weights are equal.

The advantage of the machine is that the position of each weight on its scale-pan is immaterial.

EXAMPLES. LIV.

1. A system of pulleys is such that the point of the rope to which the weight is attached descends 1 foot while the point of the rope to which the power is attached ascends 16 feet. Shew that the weight is 16 times the power (the weights of the pulleys being neglected).

2. If the weight of the pulleys in **1** is to be considered, shew that the power must be increased if the pulleys descend with the weight.

3. Two smooth inclined planes of the same height are placed back to back on the same horizontal plane. A string passes over the ridge and has a particle fastened to each of its ends; prove that if the particles are in equilibrium their masses are proportional to the slant sides of their planes.

4. A machine is constructed so that consequent on the displacement of one part of it through a foot in a certain direction another part of it is displaced the 10th part of an inch in another certain direction; the parts being connected in any manner by cog-wheels, or by strings, or by levers, or by screws, or by water; the machine being firmly fixed. Prove that a force of 1 lb. wt. applied to the first part in the direction of its possible displacement, will cause the second part to move unless there is an opposing force in the direction of its displacement of 120 lbs. wt.

5. A screw whose *pitch* (i.e. the distance between the threads) is $\frac{1}{10}$th inch is turned by means of a lever 4 ft. long; prove that the pressure necessary to raise 22 cwt. is just greater than 2·45 lb. ($\pi = \frac{22}{7}$), friction being neglected.

SECTION VIII.

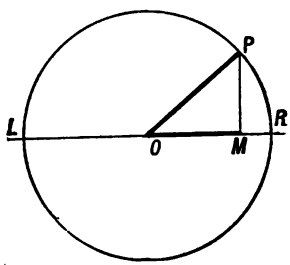
CHAPTER XXV.

SIMPLE HARMONIC MOTION.

THE PENDULUM.

317. Suppose a point P to be moving in the circumference of a fixed circle, centre O, of radius r feet, with uniform speed u velos.

Take a fixed diameter LOR. Draw PM perpendicular to LOR.



Then as P describes its circle, M moves backwards and forwards along the line ROL with a certain motion.

Familiarity with motion like that of the point M under such circumstances is of the greatest importance in physical science.

Example. Let a particle P be suspended from a fixed point by a light thread and let it describe a horizontal circle uniformly under the action of gravity.

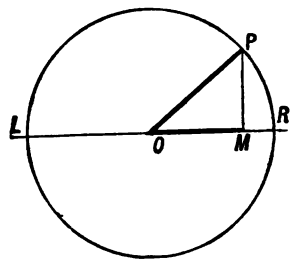
Let the motion of the particle be viewed by a spectator, standing at a distance from it which is considerable compared with the length of the string, whose eye is placed in the plane of the circle which the particle is describing.

The motion of the particle P as seen by such a spectator will be like the motion of the point M described above.

The motion of M is the motion of *each particle* of a violin string when the string is sounding a simple tone; hence

The motion of M is called a **Simple Harmonic Motion [S. H. M.]**.

The length of OR is called the **Amplitude** of the Simple Harmonic Motion.



A complete movement of M backwards and forwards is called an **Oscillation.**

The interval T seconds occupied by a *complete revolution* of P [i.e. by an oscillation of M] is called the **Period** of the S. H. M.

318. PROP. *The acceleration of a point* M *having* Simple Harmonic Motion *about the fixed point* O *is* proportional to the displacement MO *and always* towards O.

Let OR be the Amplitude of the S. H. M. and let r be the number of feet in OR.

Describe the circle RPL with centre O and radius OR.

Draw PM perpendicular to LOR.

Then as M moves in S. H. M. the point P travels with constant speed (u velos) round the circumference of the circle.

The velocity of M is in the direction OM and is the same as the resolute of the velocity of P in that direction.

Therefore the acceleration of M is the resolute parallel to OM of the acceleration of P.

The acceleration of P is $\frac{u^2}{r}$ celos along PO.

The resolute of this along MO is to $\frac{u^2}{r}$ celos as $MO : PO$.

Therefore the resolute is $\frac{u^2}{r} \cdot \frac{OM}{PO}$ celos along MO.

$$= \frac{u^2}{r^2} x \text{ celos along } MO,$$

where x is the number of feet in MO.

That is, the acceleration of M at any instant is towards O and is proportional to the length of MO. Q. E. D.

NOTE. The acceleration of P when at R, is the same as the acceleration of M when at R; hence if the acceleration of M at any point be μx celos, its acceleration at R is μr celos; whence we have

$$\frac{u^2}{r} = \mu r.$$

319. *A particle* M of m lbs. *moves in a straight line* LOR *under the action of a force tending towards the fixed point* O, *which is proportional to the displacement of* M *from the point* O; *prove that the motion of* M *is* a Simple Harmonic Motion.

Let the force be such that when M is at the distance x feet from O the particle M of m lbs. is pulled towards O with the force $m\mu x$ poundals.

Then the force acting on m lbs. at any instant is

$$m\mu x \text{ poundals.}$$

Therefore the acceleration of the m lbs. at that instant is

$$\mu x \text{ celos, towards } O.$$

By Art. 318, when a point moves in a straight line with acceleration towards a fixed point O proportional to the displacement of the point from O, then the motion of the point is S. H. M.

Therefore the motion of M is S. H. M. Q. E. D.

320. *A particle* M of m lbs. *is acted on by a force towards a fixed point* O *which varies as the distance* of M *from the point* O. *Prove that when* M *is moving in* S. M. H. *in a straight line,* **the Period is independent of the Amplitude.**

Let the force be $m\mu x$ poundals when M is x feet from O. Let the amplitude be r feet.

Then, if a point P describe a circle of radius r feet with velocity u in the same period, we have

$$\frac{u^2}{r} = \mu r.$$

The Period, T seconds, is the interval in which P describes the circumference, $2\pi r$ feet, with constant speed u velos, so that $2\pi r = Tu$.

Therefore $T = \frac{2\pi r}{u}$, $\therefore\ T^2 = \frac{4\pi^2 r^2}{u^2} = \frac{4\pi^2}{\mu}$.

Therefore the Period is $\frac{2\pi}{\sqrt{\mu}}$ seconds; which is independent of r. Q. E. D.

321. It follows that, so long as the force ($m\mu x$ poundals) acting on the particle m lbs. at the distance x feet from O is unchanged, the *period* of the oscillations is always the same, no matter what be the initial circumstances of the motion.

The oscillations of such a particle are said to be **Isochronous.**

EXAMPLES. LV.

1. Shew that when a point is moving in S. H. M. its velocity is greatest at the centre, and that its acceleration is greatest when the velocity is zero.

2. A point moving in S. H. M. is observed to pass the point distant k feet from the centre with v velos; prove that $v = u\frac{\sqrt{(r^2 - k^2)}}{r}$, where u is the speed of the guiding point of the S. H. M. and r the radius of its circle.

3. A point moving in S. H. M. is observed to pass the point distant k feet from the centre O with velocity v velos, and to pass the point distant k' feet from O with velocity v' velos; prove that the square of the amplitude is

$$\frac{v^2k'^2 - v'^2k^2}{v^2 - v'^2}.$$

4. Shew that if the angle traced out in t seconds at the centre O of its circle by the guiding point P of a point M having S. H. M. be denoted by ωt, then the period of a complete oscillation of the point is $2\pi/\omega$. And that the acceleration of the point M when distant x feet from O is $\omega^2 x$ celos.

5. A horizontal shelf is arranged so that it can only move vertically; its vertical motion is governed by that of a peg fixed to a vertical wheel of radius r feet which revolves with uniform speed u velos; shew that if u exceed $\sqrt{gr}$ an object placed on the shelf will not remain in contact with it.

THE PENDULUM.

322. DEF. A **simple pendulum** is a heavy particle suspended by a light rod or thread from a fixed point.

DEF. An oscillation is said to be **small** when the measure of its amplitude is such that the square of that measure represents a distance which is *practically* zero.

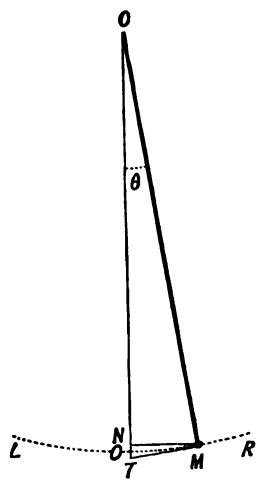
323. PROP. *To find the* Period *of the* **small oscillations** of a simple pendulum.

Let a simple Pendulum, consisting of a light rod CM and a particle M of m lbs., be performing small oscillations in the arc ROL of the circle whose centre is C.

Let $CO = l$ feet, and let O be the middle point of the arc ROL.

The oscillations are **small**; that is to say, the magnitude of the arc ROL is so small compared with CO, that the square of the fraction $\frac{RO}{CO}$ is *practically* zero.

Let M be the position of the particle at a certain instant. Draw MN perpendicular to CO and let MT be the tangent at M.



Let the angle

$$MCO = \theta = NMT.$$

At the instant under consideration the forces acting upon M are

(i) its weight mg poundals vertically downwards,

(ii) the tension of the thread along MC.

Consider the forces acting on the particle in the direction of its path; that is, along MT.

The resolute of the weight along MT is

$mg \sin NMT$ poundals† $= mg \sin \theta$ poundals

$= \frac{mg}{l} l \sin \theta$ poundals $= \frac{mg}{l} MN$ poundals.

† Or thus. The resolute along MT is to the weight as $MT : CT$; or, as $NM : CM$.

$\therefore$ the resolute $= mg \times \frac{NM}{CM}$ poundals towards N

$= \frac{mg}{l} MN$ poundals.

The tension of the string is perpendicular to the arc always and therefore produces no acceleration along the arc.

Now, so long as the chord MN is *small*, MN, MO and MT are practically equal to each other.

And therefore for **small** oscillations along the arc, the force along the arc is $\frac{mg}{l} \times MO$ poundals.

That is, the particle M moves along the arc under the action of the force, along the arc towards O, of

$$\frac{mg}{l} \times MO \text{ poundals,}$$

$$[= m\mu \times MO \text{ poundals, where } \mu = g/l]$$

which varies as the distance of M from the fixed point O.

Therefore the motion is S. H. M. and period is

$$2\pi\sqrt{\frac{l}{g}} \text{ seconds.} \qquad \text{[Art. 320]}$$

324. The above result is of great practical importance.

I. The student will notice that with two equal simple pendulums, both describing oscillations of different amplitude, both of which amplitudes are **small**, the *periods* will be equal.

Hence a slight inequality in the amplitude of the swing of the pendulum of a clock, provided each amplitude is small, does not alter the interval occupied by the swing.

II. The period of the swing, or oscillation, of a pendulum varies as the square root of its length.

Therefore an alteration in the *length* of the pendulum does alter the period of the oscillation.

325. Hence in constructing a pendulum it is important that it should represent as nearly as possible a simple pendulum whose *length* remains *constant* under different temperatures.

This is practically attained by constructing a long body whose centre of gravity is near one of its extremities, and the materials, of which it is constructed, are so arranged, that its C. G. remains at a fixed distance from that extremity from which it is suspended.

Example i. The figure represents a **Mercurial Pendulum.**

The point of suspension is *C*.

The centre of gravity is *G*.

CA is a metal rod which expands on an increase of temperature.

AB is a small glass cylinder containing mercury.

The mercury and glass also expand on an increase of temperature but the mercury expands at a *greater rate* than the glass and so rises in the cylinder bringing its C. G. nearer *C* as the temperature rises. The quantity of mercury is carefully adjusted so that the C. G. of the pendulum is practically at a constant distance from *C* for all ordinary variations of temperature.

326. NOTE. An *Oscillation* is sometimes called a double **Vibration**. So that a Vibration is a *swing* only, whilst an Oscillation is a *swing-swang*.

DEF. A **Seconds Pendulum** is a pendulum which *vibrates* once in a second.

Example i. *Find the length of the seconds pendulum.*

The period of a pendulum of length l feet is

$$2\pi\sqrt{\frac{l}{g}} \text{ seconds.}$$

[If the period is one second; i.e., if the pendulum moves backwards *and* forwards in one second we have

$$l = \frac{g}{4\pi^2} \text{ feet.}$$

In London $g = 32{\cdot}191\ldots$; $\pi = 3{\cdot}1416$.

$\therefore\ l = {\cdot}8154\ldots$feet $= 9{\cdot}7848$ inches nearly.]

If the period is two seconds, the length is

$$\frac{g}{\pi^2} \text{ feet} = 39{\cdot}1392 \text{ inches.}$$

Example ii. *Find the time of oscillation of a pendulum* 36 *inches long in London where* $g = 32{\cdot}139$.

The required interval is $2\pi\sqrt{\frac{l}{g}}$ seconds

$$= 2 \times 3{\cdot}1416\ldots \times \sqrt{\frac{3}{32{\cdot}139}} = 1{\cdot}919\ldots \text{seconds}.$$

327. It will be observed that the result of the last example depends on the magnitude of g. Hence the length of the Seconds Pendulum will be different at different places on the earth's surface. In fact, the most accurate method of obtaining the numerical value of g is by careful experiments with the pendulum.

The precautions and refinements necessary for a successful and accurate observation are beyond the scope of this work, but it may be pointed out that the period of a pendulum can be observed with great accuracy by attaching it to the works of a *clock*. For a clock is practically a machine which keeps the pendulum performing small oscillations, while it counts and records the number of such oscillations. The difficult part of the experiment is the exact measurement of the *length* of the pendulum.

328. PROP. *A pendulum* l feet *long vibrates* N *times in* T seconds; *shew that when the pendulum is lengthened* k feet *it will lose* n beats *in* T seconds *where* $n = \frac{Nk}{2l}$, *provided* k *is* small *compared with* l.

When a pendulum vibrates N times in T seconds the period of a vibration is T/N.

$$\therefore \frac{T}{N} = \pi\sqrt{\frac{l}{g}}.$$

Similarly
$$\frac{T}{N-n} = \pi\sqrt{\frac{l+k}{g}}.$$

$$\frac{N-n}{N} = \sqrt{\frac{l}{l+k}} = \left(\frac{l+k}{l}\right)^{-\frac{1}{2}};$$

$$\therefore 1 - \frac{n}{N} = \left(1+\frac{k}{l}\right)^{-\frac{1}{2}} = 1 - \frac{k}{2l} + \text{squares and higher powers}$$

of the small fraction k/l, which are neglected.

$$\therefore \frac{n}{N} = \frac{k}{2l}, \qquad \therefore n = \frac{kN}{2l}\left[= \frac{k}{2l} \cdot \frac{T}{\pi}\sqrt{\frac{g}{l}}\right]. \quad \text{Q.E.D.}$$

Example. A seconds pendulum has a bob which is moved by a screw having 50 *turns to the inch. The pendulum loses* 22 seconds per day. *Shew how to regulate it.*

Here $N = 24 \times 60 \times 60$; $n = 22$;

$$l = \frac{39{\cdot}14}{12} \therefore k = \frac{22 \times 39{\cdot}14 \times 2}{24 \times 60 \times 60 \times 12} = \frac{{\cdot}02}{12} \text{ nearly};$$

$\therefore$ the bob must be raised ·02 inches, or, one complete turn of the screw.

329. PROP. *To find the number of* vibrations *gained in* T seconds *when a pendulum is moved from one place to another.*

Let g and $g+g_1$ be the accelerations due to gravity at the first and second places respectively. Then g_1 is small compared with g. Then as in the last article

$$\frac{T}{N} = \pi\sqrt{\frac{l}{g}}, \quad \frac{T}{N+n} = \pi\sqrt{\frac{l}{g+g_1}};$$

$$\therefore \quad \frac{N+n}{N} = \sqrt{\frac{g+g_1}{g}} = \sqrt{1+\frac{g_1}{g}} = 1 + \tfrac{1}{2}\frac{g_1}{g} + \text{ squares and higher powers of } \frac{g_1}{g}, \text{ which are neglected.}$$

$$\therefore \quad \frac{n}{N} = \tfrac{1}{2}\frac{g_1}{g}.$$

Example. A pendulum which beats seconds at Greenwich is carried to Paris where $g = 32{\cdot}18$. How much will it lose per day?

Here $N = 86400$, $g = 32{\cdot}19$, $g = {\cdot}01$,

$$\therefore \quad n = \frac{1}{2} \cdot \frac{86400}{32{\cdot}19} \times {\cdot}01 = \frac{43200}{3219} = \text{about } 13;$$

$\therefore$ the pendulum will (other things being equal) lose 13 seconds per day.

EXAMPLES. LVI.

1. Find the length of the seconds pendulum in Paris ($g = 32{\cdot}182\ldots$).

2. Find the length of the pendulum which vibrates once in two seconds in London ($g = 32{\cdot}19\ldots$).

3. Find the length of the pendulum which oscillates once in a second at the equator ($g = 32{\cdot}09$).

4. A pendulum carried from the sea level to the top of a mountain loses n oscillations in N oscillations. Shew that the height of the mountain is to the radius of the earth as $n : N$ nearly. [N.B. The acceleration due to gravity varies inversely as the square of the distance of the place from the centre of the earth when the place is *outside* the earth.]

5. A pendulum carried from the sea level down a mine is observed to lose n oscillations in N oscillations, prove that the depth of the mine is to the diameter of the earth as $n : N$ nearly. [N.B. The acceleration due to gravity varies directly as the distance of the place from the centre of the earth when the place is *inside* the earth.]

6. A seconds pendulum when carried to the top of a mountain loses a half a minute per day, what is the height of the mountain? [Radius of earth = 4000 miles.]

7. A seconds pendulum loses 3 seconds a day when taken to the bottom of a mine; shew that the mine is about 1450 feet deep.

8. Find the length of a pendulum at London which beats 3000 times an hour.

9. A clock whose pendulum is a foot long loses 100 seconds a day; how must you alter the pendulum?

10. A clock whose pendulum is 10 inches long gains a minute a day; how can the error be corrected?

11. If a clock which keeps correct time at the equator gains 5 minutes per day when taken to the pole (all other circumstances being supposed equal), find the value of g at the pole. [g at equator = 32·09.]

12. A seconds pendulum in consequence of a change of temperature has its length diminished by a inches, and consequently gains n seconds per day; find the length of the pendulum.

13. A man finding that his clock was gaining, placed some pieces of lead round the rod of the pendulum above the bob, thereby increasing its weight; did he make the clock go slower?

14. In London $g = 32{\cdot}19\ldots$ celos; in Paris $g = 32{\cdot}1825\ldots$ celos; in New York $g = 32{\cdot}16\ldots$ celos; prove that the lengths of the seconds pendulum at these places are $39{\cdot}139\ldots$, $39{\cdot}129\ldots$, $39{\cdot}10\ldots$ inches respectively.

15. A small heavy ball, suspended by a fine thread 6 ft. $4\frac{1}{2}$ inches long, is observed to make 151 oscillations in 3 min. 31 secs.; what is the accelerative effect of gravity at that place?

16. If u velos is the velocity of the bob of a simple pendulum at its lowest point, then its velocity t seconds after passing through its lowest point is $u \cos(2\pi t/P)$, where P is its period for small oscillations.

330. It can be shewn (but the proof is not simple enough to be inserted here) that any rigid body capable of motion only in one plane, can so far as its dynamical properties are concerned be replaced by two particles which must fulfil the following conditions.

I. They must have the same total mass as that of the body.

II. They must have the same centre of gravity as the rigid body.

III. They must be connected by a light rigid rod and must be so arranged that if their C. G. and that of the rigid body be each fixed and they are each acted on by equal forces tending to turn each of them about their C. G. their rotation is exactly the same.

It is found that these conditions can be satisfied when any point whatever is taken as the position of one of the particles.

331. Now let us suppose that we wish to consider the motion of a rigid body AB of mass M about some axis fixed in the body. Such a body can move only in one plane.

Let us replace the body by the two particles P, Q which are dynamically equivalent to it, *one of which is placed in the axis*, at the point P, and is of mass m. Let G be the C. G. of the body.

Then the other particle Q is in the line PG produced and is of the mass m' such that

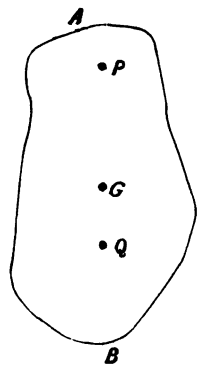
$$m + m' = M.$$

Then for all movements about the axis through P the body AB may be replaced by the particles P and Q united by a light rigid rod.

Hence if the body AB is allowed to perform small oscillations under the action of gravity it will move exactly as a simple pendulum of length PQ.

Hence PQ is called the length of **the simple equivalent pendulum** of the rigid body AB corresponding to the axis of suspension through P.

332. Moreover for all motion in the plane perpendicular to the axis through P, the two particles P, Q have the same properties as AB. Hence if the body be suspended under the action of gravity from an axis through Q it will move just as the two particles PQ connected by a light rigid rod, the particle Q being fixed. That is, it will oscillate as a simple pendulum of length P, Q.



333. When a rigid body, suspended from a fixed point C, oscillates as if it were a mass concentrated at P, the point P is called the **centre of oscillation** with respect to the centre of suspension C.

334. Art. 332 proves that the **centre of suspension and the centre of oscillation are interchangeable.**

Now by Art. 331 we may, for the purpose of considering how a rigid body will behave under the action of force, replace the body by another consisting of two particles $P+Q$ connected by a light rigid rod, having equal mass and the same C. G. as the body, and if the masses of P and Q are properly adjusted we may put *one* of the particles at any point of the body which we wish.

When a rigid body, consisting of two particles $P+Q$ connected by a light rigid rod, and under the action of no forces, is struck at the point P, by an impulse in the direction perpendicular to PQ, the immediate effect is to give the body an angular velocity about Q.

Example. Suppose a heavy mass suspended from a fixed centre of suspension, and suppose that a horizontal impulse be given to the mass in the plane in which it is free to turn, at the corresponding centre of oscillation. The mass would behave just as the particle of a simple pendulum receiving a horizontal impulse.

Hence two corresponding centres of oscillation and suspension are sometimes called a **centre of percussion** corresponding to **an instantaneous centre of rotation.**

335. The student must notice that any point may be chosen as one of a pair of centres of suspension and oscillation and that then the other is fixed and that the line joining them always passes through the C. G. of the rigid body and that the C. G. lies *between* the pairs of centres.

KATER'S PENDULUM. Take a uniform metallic rod, and fix to it two symmetrical knife-edges, or axes, on either of which it can swing as a pendulum. Let these knives be fixed, one at any chosen point C near one end, and the other near Q the centre of oscillation corresponding to the centre of suspension C at the first knife-edge. And let these knife-edges be capable of adjustment by means of properly arranged screws. The rod must now be set up as a pendulum and its period observed when swinging on each knife-edge in turn; and the knife-edges adjusted until the periods of the rod are the same for each knife-edge.

When this has been done the knife-edges occupy the positions C and Q of corresponding centres of suspension and oscillation; and the distance between them gives the length of the simple equivalent pendulum. The distance l can be measured, and the period T observed and then g can be found from the formula of Art. 323, $T = 2\pi\sqrt{\frac{l}{g}}$.

ANSWERS.

I.

1. $27\frac{3}{11}$ velos. **2.** 12 min. $34\frac{2}{7}$ sec.

3. (i) $27\frac{5}{11}$ miles per hour. (ii) $3\frac{9}{22}$ miles per hour.

4. (i) $\frac{11}{3600}$ velos. (ii) $\frac{11}{9072}$ velos. **5.** $1536\frac{12}{13}$ velos.

6. 1 min. $1\frac{1}{13}$ sec. **7.** 500 yards in $11\frac{1}{2}$ sec.; their ratio is 66 to 100.

8. 11 to 6.

9.
(i) 30 velos; $20\frac{5}{11}$ miles per hour.
(ii) $26\frac{4}{13}$ velos; $18\frac{1}{13}$ miles per hour.
(iii) $20\frac{10}{121}$ velos; $14\frac{109}{121}$ miles per hour.
(iv) $17\frac{111}{1111}$ velos; $11\frac{1111}{1111}$ miles per hour.

II.

1. $\frac{22c}{15}$ velos. **2.** $\frac{15hv}{22}$ miles. **3.** $\frac{3m}{k}$ seconds.

4. $\frac{45hm}{22t}$ miles. **5.** $40\lambda k$ *yards*. **6.** $\frac{22mt}{15k}$ hrs. **7.** $\frac{22nt}{45k}$ yards.

III.

1. $10\frac{1}{2}$ velos. **2.** (i) 10 velos. (ii) 11 velos. **4.** 4 velos; $33\frac{1}{3}$ yds.

5. $\frac{5ab}{2b}$ velos. **6.** $9v$ velos.

IV.

1. 13 velos. **2.** 55 velos.

3. 255 velos, 210 velos, 165 velos, 120 velos. **4.** 65 velos.

5. 277 velos. **6.** 3 min. 18 sec. (after it begins to decrease)

7. (i) 8 secs. from rest. (ii) 1 m. $10\frac{2}{3}$ sec. **8.** 40 sec. previously.

9. 4 sec.; 8 sec. **10.** 11 sec. **11.** 96 ft.; 48 ft.; 96 ft.

12. $3\frac{1}{2}$ sec. previously; 7 sec. previously.

VI.

1. (i) 102 feet. (iii) 12,000 feet or $2\frac{8}{11}$ miles.
(ii) 208 feet. (iv) 27,600 feet or $5\frac{5}{22}$ miles.

5. 10 seconds before the 5 sec. **6.** Yes. **7.** 320 velos.

8. 81 ft. **9.** $48\frac{3}{4}$ feet. **10.** 32 ft.; $2\frac{1}{2}$ secs. previously.

VII.

1. (i) 80 velos. (iv) 6720 ft.
(ii) 176 velos. (v) in $33\frac{7}{8}$ sec.
(iii) 320 ft. (vi) half a sec. previously.

2. 20 celos. **4.** 640 feet. **5.** 4 celos.

6. (i) 32 celos. (ii) 144 feet.

7. (i) 15 velos. (iv) 3 to 1.
(ii) 6 sec.; 90 ft. (v) $(6-3\sqrt{2})$ sec. $= 1{\cdot}75\ldots$ secs.
(iii) $87\frac{1}{2}$ ft.

8. (i) -32 celos.
(ii) in $4\frac{1}{2}$ sec. from the commencement of the first interval.
(iii) -196 ft. from its position when at rest.
(iv) it starts with 144 velos.
(v) in 9 sec. from commencement of first interval.

9. $\frac{k-2h}{3t^2}$ celos. **10.** $\frac{k-h}{2t+4}$.

VIII.

1. 3 sec. **2.** (i) $10(\sqrt{2}-1)$ sec. $= 4{\cdot}14$ sec. (ii) $56\frac{1}{4}$ ft.

3. 100 sec.; 180000 ft. **4.** 40 sec.; 25600 ft.; in 20 secs.

5. 6 celos. **6.** 2694 velos (nearly). **7.** $\frac{11}{16}$ celos.

8. $-3{\cdot}3$ celos. **9.** 2640 ft. **10.** $\frac{1331}{2197}$ celos.

11. It has $-\frac{5}{11}$ celos, therefore the required interval t secs. is one of the roots of the equation $300 = 22\frac{5}{11}t - \frac{1}{2}\times\frac{5}{11}t^2$, that is $t = 49 \pm 32{\cdot}88\ldots$secs.

12. It has $-\frac{5}{48}$ celos; t is one of the roots of $400 = 13\frac{3}{4}t - \frac{1}{2}\times\frac{5}{48}t^2$; $t = 134 \pm 101{\cdot}3\ldots$ **13.** In 4 secs. from the first instant.

14. $\frac{9}{16}$ celo. **15.** 64 ft.; 64 velos. **16.** 54 celos. **17.** 1 celo.

18. 16 celos; 36 velos. **19.** Their ratio is 1 to 4 [$v^2 = 2as$].

20. 256×2 ft.

IX.

1. $\frac{1}{2}$ celo. **2.** 32 celos. **3.** $\frac{1}{2}$ celo. **4.** $12\frac{1}{2}$ celos.
5. $\frac{3}{7}$ celo. **6.** $\frac{1}{64}$ celo. **7.** 5 velos; 25 ft.
8. 48 velos; 96 ft. **9.** 2 velos; 3 ft. **10.** $5\frac{1}{7}$ velos; $23\frac{1}{7}$ ft.
11. 3 velos; 18 ft. **12.** 16 velos; 4 ft.
13. (i) 8 velos; 18 ft. (ii) 22 velos; 84 ft. (iii) 158 velos; $283\frac{1}{2}$ ft.
14. 120 ft. **15.** 512 ft.
16. after 2 sec.; 32 velos; or, after 4 sec.; -32 velos.
17. after 2 sec.; and after 3 sec. **18.** 54 poundals.
19. 32 poundals. **20.** 36 velos; 320 poundals.
21. 20 lbs. **22.** $\frac{49}{11}$ lbs.
23. In the first minute it travels $6\frac{1}{4}$ ft.; it travels $14\frac{7}{13}$ ft. further before coming again to rest. **24.** One is four times as great as the other

X.

1. 96 poundals. **2.** 1 poundal. **3.** $20 \times 2240 \times 32$ poundals.
4. $4 \times 112 \times 32$ poundals. **5.** 4 lbs. wt. **6.** $31\frac{1}{4}$ lbs. wt.
7. 100 lbs. wt. **8.** 14 lbs. wt. **9.** 125 lbs. wt.
10. 4×112 lbs. wt. **11.** 2 celos. **12.** 110 secs.
13. 480 velos. **14.** $\frac{4 \times 100 \times 2240}{32}$ lbs. weight $= 28000$ lbs. weight.
15. $\frac{28}{15}$ lbs. weight. **16.** 14 m. 40 sec. **17.** In 50 secs.
18. 308 secs. **19.** 8×45 tons wt. **20.** 64 lbs.

XI.

1. 4 sec. **2.** (i) 48 ft. (ii) 112 ft. **3.** 10 sec.; 1600 ft.
4. 64 velos; 64 ft. **5.** In 4 sec. from time stone is thrown
6. 2 sec.; 192 ft. from the foot of the tower.

XII.

1. $\frac{16}{100} = \frac{4}{25}$ celos. **3.** Their ratio is 20 to 21. **4.** $\frac{18}{11}$ ft.
5. $\frac{64}{121}$ ft. **6.** $4 \times 14 \times 22$ poundals.

XIII.

1. $300 \times 20 \times 112$ poundals.
2. $250 \times 20 \times 112$ poundals; $100 \times 20 \times 112$ poundals;
$200 \times 20 \times 112$ poundals; $50 \times 20 \times 112$ poundals.
$150 \times 20 \times 112$ poundals;
3. $500 \times 20 \times 112$ poundals.

4. (i) $\frac{220\times 20\times 112}{3}$ poundals. (iii) $\frac{22\times 20\times 112}{3}$ poundals.
(ii) $44\times 20\times 112$ poundals. (iv) 176 yards.

5. (i) $\frac{8a}{15}$ celos. (ii) $\frac{7ma}{15}$ poundals. (iii) $\frac{ma}{5}$ p. (iv) $\frac{ma}{15}$ p.

6. (i) $\frac{10a}{10+n}$ celos. (ii) $\frac{n-4}{10+n}p$ poundals.

XIV.

1. $\frac{25}{8}$ poundals; 2 celos. **2.** 120 poundals; 8 celos.
3. $\frac{57}{13}$ poundals; $\frac{1}{3}$ celos. **4.** $\frac{25}{2}$ poundals; 2 celos.
6. $\frac{1}{2}$ a second. **7.** $30\frac{15}{16}$ ozs. **8.** $15\frac{1}{2}$ ozs. **9.** $12\frac{5}{16}$ lbs.
10. 17 : 15. **12.** $\frac{f-f'}{m+m'}$ celos.
14. Velocity before : velocity after $=\sqrt{3}:\sqrt{2}$.
15. 22×112 poundals. **16.** The weight of $1\frac{1}{2}$ cwt. **18.** 3600 ft.
19. $\frac{8}{81}$ of the weight of 2 ozs. **20.** $1\frac{17}{18}$ of the weight of 2 ozs.

XV.

1. $44\times 100\times 2240$ pulses. **2.** $\frac{1}{3}\times 44\times 2240$ poundals $=1026\frac{2}{3}$ lbs. wt.
3. 3 hrs. $3\frac{1}{3}'$. **4.** $264\times 10000\times 224$ pulses; $27\frac{1}{2}$ tons weight.
5. 30 times $27\frac{1}{2}$ tons wt. $=825$ tons wt. **6.** $22\times 8000\times 2240$ pulses; $18\frac{1}{2}$ tons wt. **7.** $88\times 50\times 2240$ pulses; 275 tons wt. **8.** 10 pulses; (i) 10 oz. wt. (ii) $3\frac{1}{8}$ lbs. wt. (iii) $312\frac{1}{2}$ lbs. wt.

XVI.

1. 10 velos. **2.** $\frac{mv}{m+m'}$ velos. **3.** 9 velos. **4.** $\frac{8}{15}$ velos.
5. $5\frac{1}{2}$ velos. **6.** $22\frac{1}{2}$ velos. **8.** 500×2500 poundals.

XVII.

1. $15\frac{1}{3}$ velos; $19\frac{1}{3}$ velos. **2.** $-2\frac{1}{3}$ velos; $5\frac{1}{3}$ velos.
3. $-\frac{1}{3}v$; $5\frac{1}{3}v$. **4.** 8 velos.
6. $\left(\frac{e+1}{2}\right)v$; $\left(\frac{e+1}{2}\right)^2 v$; $\left(\frac{e+1}{2}\right)^3 v$; $\left(\frac{e+1}{2}\right)^n v$.
7. In 1 sec.; 16 ft. below the point from which m' is let fall;

$$v=-\frac{32}{m+m'}\{m'+2em'-m\};\quad v'=\frac{32}{m+m'}\{m+2em-m'\},$$

v and v' being measured upwards.

8. $\frac{1}{1+2e}$. **9.** Twice the time occupied in falling 4 ft. from rest.
10. $\sqrt{\frac{1}{3}}$.

XVIII.

1. $\frac{1}{10}$ velo; $\frac{1}{12000}$ celo. **2.** $\frac{22}{15}$ velo; $\frac{121}{270000}$ celo.

3. $\frac{3a}{b}$ velos; $\frac{3a}{b^2}$ celos. **4.** $\frac{5}{88}$ sec. **5.** 25 ft.

6. $32\sqrt{330}$ velos. **7.** $\frac{1000}{11}$ sec.; $(\frac{40}{11})^2 + 400$ feet. **8.** 50 ft.

9. 32 lbs. **10.** $\frac{20 \times 112 \times 5280}{(60)^4}$ poundals. **11.** (i) ·0328... velos.

(ii) ·0328 celos. (iii) (·0328...) × (·0022...) poundals.

13. In Paris a lb. weighs 32·18... pdls. = 32·18... × 13825 dynes.
New York ..... 32·16... .....
Equator ..... 32·09... .....
Pole ..... 32·25... .....
and a gramme = ·0022046... lbs.

14. The unit mass is that which when acted on by the unit force has unit acceleration. The unit acceleration in this case is 1 celo. The weight of a pound is g poundals; g poundals produce 1 celo in g pounds; hence the unit mass is, at Equator, 32·09... lbs.; in Paris, 32·18... lbs.; in London, 32·19... lbs.; in New York, 32·16... lbs.

XIX.

1. 500 yards. **2.** 5 miles. **3.** 10·2 ft. nearly. **4.** 25 yards.

5. 103·6 ft. **6.** 14 feet; 2 feet; 10 feet; 13 feet nearly.

7. 350 yards; 60° with first direction.

8. 115 yds. nearly; making an angle $\tan^{-1}\frac{\sqrt{2}}{10+\sqrt{2}}$ East of North.

9. $20\sqrt{3}$ yards.

10. 3·6 miles, in a direction making the angle $\tan^{-1}\frac{1}{8}$ with the North.

XX.

1. 50 yards; 86·5... yards. **2.** 15 yards, 20 yards.

3. 323·8 yards. **4.** $\sqrt{\{(662\cdot13)^2 + (212\cdot13)^2\}}$.

5. ·707... mile; 0 miles. **6.** 180 yards. **7.** 2·9... miles.

8. 6·38... miles.

XXI.

1. 150 feet from its initial position; in a direction making an angle whose sine is $\frac{4}{5}$ to the East of North.

2. $\sqrt{\{(30)^2 + (45)^2\}}$ feet from its initial position.

3. 24 ft. south of its initial position.

4. $8\sqrt{\{(35)^2+15)^2\}}$ feet from its initial position.
5. $\sqrt{\{(200)^2+(150)^2\}}$ feet from its initial position.
6. The point is at rest.
7. $\sqrt{\{(40)^2+(64)^2\}}$ feet from its initial position.
8. 10 feet to the north of its initial position.

XXII.

1. (i) horizontal distance 100 ft., vertical distance 16 ft. below the point of projection; (ii) horizontal distance 200 ft., vertical distance 64 ft. below the point of projection; (iii) horizontal distance 300 ft., vertical distance 144 ft. below the point of projection.
2. Draw OQ at an angle of 45° to the horizon, and then from Q draw QP vertically downwards so that (i) $OQ=100$ ft., $QP=16$ ft.; (ii) $OQ=200$ ft., $QP=64$ ft.; (iii) $OQ=300$ ft., $QP=144$ ft.
3. Draw OQ at an angle 30° to the horizon and then from Q draw QP vertically downwards so that (i) $OQ=1000$ ft., $QP=16$ ft.; (ii) $OQ=2000$ ft., $QP=64$ ft.; (iii) $OQ=3000$ ft., $QP=144$ ft.
4. 960 velos. 5. 225 ft. below the top.
6. $\sqrt{\{(16)^2+(11)^2\}}$ feet = about 19·4 feet.

XXIII.

1. 20 velos. 2. 25 velos. 3. 50 miles per hour $=73\frac{1}{3}$ velos. 4. 8·2... velos.
5. $\sqrt{a^2+b^2}$ velos. 6. $\sqrt{a^2+\left(\frac{22m}{15}\right)^2}$ velos. 7. 3 velos.
8. $a\sqrt{3}$ velos. 9. 6·7... velos. 10. $10\sqrt{7}=26\cdot4...$ velos.
11. 20 velos. 12. 42·7 velos. 13. 57·46 ft.; see Ex. i. p. 84.
14. 35·35... velos, 35·35... velos. 15. 99·6 velos.
16. 1136 velos nearly. 17. 28·28... velos. 18. 120° from N.
19. It has received $(2-\sqrt{2})^{\frac{1}{2}}$ times its former velocity, making an angle $\cos^{-1}\frac{1}{2}(2-\sqrt{2})^{\frac{1}{2}}$ with it. 20. 67 velos nearly. 21. 30 velos.
22. 20·5 velos making angle with horizon $\tan=\frac{2}{3}$.
23. At a point $\sqrt{(20)^2+(\frac{28}{3})^2}$ ft. from point of stoppage.
24. 21 velos nearly.

XXIV.

1. Angle in direction of train whose $\tan=-\frac{3}{4}$. 2. 13 celos.
3. 13 velos. 4. 50·4... velos. 5. 44·78 velos nearly.
6. 135° with direction of train.

XXV.

1. Resultant 5 lbs. wt., angle 53°. 7′ nearly. [These angles are found from a table of tangents.]
2. 18·02... pdls., 33°. 40′.
3. 65 lbs. wt., 67°. 23′.
4. 9·219 pdls., 49°. 24′.
5. 14·86 lbs. wt., 47°. 42′.
6. 22·36 pdls., 63°. 26′.
7. 5·657 lbs. wt., 45°.
8. 2 pdls., 60°.

[The angle in each of these cases is that which the resultant makes with the direction of the first of the forces given.]

XXVI.

1. $4\sqrt{5}$ pdls.
2. $3\sqrt{7}$ lbs.
3. $\sqrt{313}$ lbs.
4. $\sqrt{(25+12\sqrt{3})}$ lbs.
5. $\sqrt{97}$ pdls.
6. $5\sqrt{(5+2\sqrt{2})}$ lbs.
7. 5 lbs.
8. 10 lbs.
9. $10\sqrt{(13+6\sqrt{3})}$ lbs.
10. $10\sqrt{(13-6\sqrt{3})}$ lbs.
11. $10\sqrt{509}$ lbs.
12. P lbs.
13. $\sqrt{28}$ lbs.
14. $\sqrt{6}$ lbs.
16. ·6 seconds.
17. 4·4 secs.
18. It will be $20\sqrt{2}$ feet to the N.W. of its original position and it will have $8\sqrt{2}$ velos.
19. The direction of motion makes the angle whose tangent is $\frac{3}{4}$ with the force of 4 cwt.; 8 celos.
20. $4\frac{1}{3}$ celos.

XXVII.

1. 15 pdls.
2. 43·3 lbs. wt.
3. 4 lbs. wt.
4. 75 lbs. wt.
5. 210 lbs. wt.
6. 54·625 lbs. wt.
7. 188·178 lbs. wt.
8. 20·2995 lbs. wt.
9. 20 lbs. wt.
10. 2·1 lbs. wt.
11. $16\frac{3}{4}$ lbs. wt.
12. $7\frac{3}{4}$ lbs. wt.
13. 16·9 lbs. wt.
14. 9 lbs.

XXVIII.

1. 6·48... lbs.
2. 5 lbs.
3. 7·07... lbs.
4. 20 lbs.
5. 2 lbs. : 1 lb.
6. 20 lbs.
7. 9 lbs.
12. 5 lbs.; −5 lbs.

XXIX.

1. 25; $15\sqrt{2}=21·2...$; zero.
2. 12·5; zero; 17·5.
3. 25; $\frac{1}{2}\sqrt{2}=10·6...$; 17·5.
4. 72; 72; 30; zero.
5. 72; 24; 30; 144.
6. 7 inches from A.
7. 42 inches from A.
8. The point is $\frac{5}{8}$ of BC from B.

XXXI.

1. The resultant of the forces cuts the line joining these two points in the ratio of 2 to 1.
2. The resultant, if not zero, must pass through Q.

XXXIII.

1. 5 lbs. acting 3 ft. from smaller force.
2. $2\frac{1}{2}$ ft. from smaller force.
3. 64 lbs.; 48 lbs.
4. 48 lbs.; 32 lbs.
5. 12 lbs.; 24 lbs.
6. 24 lbs.; 16 lbs.
7. An unlike parallel force of 10 lbs. acting at the middle point.
8. An unlike parallel force of 3 lbs. acting at a point $2\frac{3}{8}$ ft. from the other unlike force.
9. 3 cwt.
10. $7\frac{1}{2}$ ft. from fixed point.
11. An unlike parallel force of 21 lbs. acting at a point 4 ft. from smallest weight.
12. An unlike parallel force of 7 lbs. acting at a point 6 ft. from the smaller weight.
13. 2 ft. from the stronger man.
14. 4 ft. from him.
15. $267\frac{3}{4}$ lbs.; $624\frac{1}{3}$ lbs.
16. $7\frac{5}{19}$ ft.; $12\frac{3}{19}$ ft.
17. 8 lbs. and 6 lbs.
18. $116\frac{2}{3}$ lbs.
19. 350 lbs.
20. 42 lbs.; 21 lbs.
21. $17\frac{1}{2}$ ft.
22. 96 lbs.
23. 96 lbs.
24. 21 lbs. acting at a point $3\frac{1}{3}$ in. from end.
25. 27 lbs. ..... $1\frac{2}{7}$ ft. .....
26. 24 lbs. ..... $5\frac{1}{3}$ ft. .....
27. 8 ft. .....
28. $2\frac{4}{15}$ ft. from end; 1 cwt., 2 cwt., 4 cwt., 8 cwt. respectively.

XXXIV.

1. OM in line $OC = 2\frac{1}{12}$ in., $ON = 2\frac{3}{8}$ in.
2. $OM = 2$ in., $ON = 1\frac{1}{16}$ in.
3. $OM = 1$ in., $ON = \frac{1}{2}$ in.
4. $DM = 3\frac{3}{4}$ in., $CN = 2{\cdot}165\ldots$ in.
5. The point of intersection of the diagonals.
6. The C.G. of the triangle. See p. 136.
7. At the centre of the Inscribed Circle. $CM = s - c$, $CN = \frac{S}{s} = r$, see *Elementary Trigonometry*, p. 232.
8. If the force at C be unlike that of the other two then
$$CM = a + b \cdot \cos c, \quad CN = b \sin c.$$
If the parallelogram $ACBO$ be completed, O is the centre of the forces.

9. At the centre of an Escribed Circle. If the force at C be the unlike force, then $CM=s$, $CN=r_3$, see *Trigonometry*, p. 233.

10. CM in direction $CD=\frac{1}{2}(2c-d.\cos D+b\cos C)$,
CN in direction perpendicular to $CD=\frac{1}{2}(b\sin C+d\sin D)$,
$AB=a$, $BC=b$, $CD=c$, $DA=d$.

XXXV.

1. $\frac{1}{4}$ of length of rod from point of suspension of 10 lbs.
2. $\frac{1}{12}$ of length of rod from point of suspension of 3 lbs.
3. 24 lbs., 20 lbs. **4.** 8 ft. from the other end.
5. $28\frac{1}{4}$ ft. from the first man.
6. $5\frac{3}{8}$ ft. from point of suspension of 8 lbs.
7. $6\frac{3}{5}$ ft. from point of suspension of 12 lbs.
8. 10 ft. from that end. **9.** 6 lbs. **10.** 10 lbs.
11. $1\frac{5}{7}$ ft. from point of suspension of 7 lbs.
12. $3\frac{1}{4}$ ft. from point of suspension of 1 lb.
13. $2\frac{1}{8}$ ft. from point of suspension of 3 lbs.
14. 16 lbs. **15.** 2 lbs. **16.** 8 lbs.
17. $267\frac{3}{7}$ lbs.; $624\frac{1}{7}$ lbs. **18.** $7\frac{3}{11}$ ft.; $12\frac{10}{11}$ ft. **19.** 16 lbs.
20. $33\frac{1}{3}$ lbs. **21.** $8\frac{1}{2}$ lbs. **22.** 6 lbs. **23.** 15 ft.

XXXVI.

1. C.G. is $2\frac{1}{8}a$ from the foot of the cross [a=side of square].
2. C.G. is distant $\frac{2}{5}$ of the diameter measured from the point of suspension of 2 lbs.
3. C.G. is distant $\frac{1}{7}$ of the diameter measured from the point of suspension of 1 lb.
5. C.G. is $5\frac{7}{8}$ in. from the middle point of the lowest side of the figure.
6. C.G. is distant $\frac{1}{3}\frac{2}{3}$ of the diameter from that corner.
7. C.G. is distant $\frac{3}{8}$ of the perpendicular from E on DC.
8. C.G. is distant $\frac{3}{5}$ of the line drawn from the middle of base to vertex.
9. C.G. is distant $\frac{1}{15}$ of the radius of the greater circle from its centre.
10. C.G. is distant $\frac{1}{12}\sqrt{2}$ of the radius bisecting the angle between the two radii from the centre.
11. C.G. is distant $\frac{1}{3}$ of the radius drawn from centre to B.
12. C.G. is distant $\frac{1}{3}$ of the line drawn from centre to the middle of AD.
13. C.G. is distant $\frac{1}{3\sqrt{2}}$ of OD.
14. As $\sqrt{3}:1$.

16. C.G. is distant $\frac{5}{12}$ of the line drawn from the middle of the opposite side to the other angle.

17. C.G. is distant $\frac{3}{8+2\sqrt{3}}$ of the base from its middle point.

19. C.G. is distant $\frac{11}{16}$ of the side of the square from the middle point of the base.

20. Let the C.G. be at G; draw GM, GN perpendiculars to DA, DC; then $GM=\frac{11}{16}$ of DC; $GN=\frac{5}{8}$ of AD.

21. C.G. is distant from E $\frac{1}{12}$ of the length of the line joining the middle of DC with E.

28. The tangent of each base angle is 3. **29.** 3.

XXXVII.

2. Two-thirds of the line joining the vertex with the middle point of the opposite side. **3.** 45°, 135°.

5. $a\sqrt{3}$. [a=side of square.] **7.** 24·142... lbs.

9. 120 lbs. **10.** 18 lbs.

11. 17 bricks; for the C.G. of 17 bricks is vertically over the edge of the 18th brick.

XXXVIII.

1. A force of −1 lb. **2.** A force of −11 lbs. **3.** A force of −3 lbs.

4. AC must be produced to point Q, so that $AQ=18$ inches.

5. 18 in. from E in the same direction.

6. 1·519... inches. **7.** $1\frac{1}{3}$ in. from A.

8. (i) AD must be produced 7 in.
(ii) DC must be produced $1\frac{1}{2}$ in.

9. It cuts AD produced to M so that $AM=41{\cdot}2$ in., and DC produced to N so that $DN=12{\cdot}6$ in.

11. A force of 6 lbs. acting in a direction parallel to BC.

12. The resultant of the three forces is parallel to this perpendicular; so that no such point can be found.

14. $2\sqrt{3}$ lbs. at B, $\frac{1}{2}\sqrt{3}$ lbs. at A.

15. A weight equal to $2Q$ must be suspended from B.

17. 2 lbs. **18.** $\sqrt{3}$ lbs. **19.** 100 lbs.

XXXIX.

1. 141 lbs. **2.** 116 lbs. **3.** A weight of 30 lbs.

5. (i) A force of 20 lbs. (ii) 100 lbs.

6. When the end of the chain has reached a point 20 feet from the ground. **8.** 5 ft. 4 in.; 7 ft. 4 in.

9. (i) the 4 lb. weight is at the lowest point; (ii) 7 lbs., 5 lbs.

10. (i) the 10 lbs. weight is at the lowest point. (ii) The weight of 10 lbs. is at its lowest point and one part of the chain supports 7 lbs. more than the other; therefore the tensions of the two parts of the chain at the lowest point are $8\frac{1}{2}$ lbs. and $1\frac{1}{2}$ lbs. (iii) $16\frac{1}{2}$ lbs.

11. 9 lbs. **12.** 20 lbs. **13.** 5 lbs.

14. 10 lbs. **15.** $67\frac{1}{5}$ lbs.; $50\frac{3}{5}$ lbs. **16.** $78\frac{6}{13}$ lbs.; $188\frac{4}{13}$ lbs.

17. $50\frac{3}{5}$ lbs.; $67\frac{1}{5}$ lbs. **18.** $188\frac{4}{13}$ lbs.; $78\frac{6}{13}$ lbs.

XL.

1. 40 lbs. **2.** $3\frac{1}{3}$ ft. **3.** $7\frac{1}{7}$ lbs. **4.** $2\frac{1}{2}$ lbs.

5. Any distance less than $3\frac{3}{4}$ ft.

6. $4\frac{8}{17}$ inches. **7.** $2\frac{1}{13}$ lbs. **8.** 12 lbs.

9. (i) Place the weight at one end with an arm of 5 ft. and the power at the other with an arm of 10 ft.

(ii) Place the weight $7\frac{1}{2}$ ft. from the fulcrum which is at one extremity and the power acting vertically upwards at the other end.

11. $1\frac{1}{7}$ ft. **12.** $11\frac{1}{3}$ ft.

13. (i) $\sqrt{9} : \sqrt{10}$; (ii) $6\sqrt{10}$ lbs. $= 18{\cdot}97\ldots$ lbs.

14. $9\frac{1}{3}$ lbs. **15.** (i) $18\frac{3}{5}$ lbs. (ii) $\frac{1}{5}\sqrt{\{49^2+46^2\times 3\}}$ lbs.

16. Q must be multiplied by $\frac{P^2}{Q^2}$.

18. 40 lbs. **19.** 90°; 20 lbs. **20.** 26·7 lbs. **21.** 1 lb.

XLI.

2. The grocer loses $\frac{1}{2}\frac{(a-b)^2}{ab}$ of the nominal weight, where a, b are the lengths of the arms. **3.** 13 in.; 14 in. **5.** $10\frac{5}{13}$ lbs.

7. 1 oz. **11.** He loses $\frac{1}{4}d.$ on every 2 lbs. **12.** As 11·3 : 16.

XLII.

1. 5 lbs.; 1 inch. **2.** The nominal weight is always 5 lbs. too great.

3. 18 in. **4.** 26 lbs.; 15 lbs.: 2 in. **6.** $5\frac{1}{3}$ in. from fulcrum; 4 in.

8. At the fulcrum. **9.** The fulcrum is 8 in. from the C.G. of the beam.

11. C.G. is distant 12 inches from the end to which the weight is attached: 4 oz. **12.** 36 lbs.

XLIII.

1. 1 lb. **2.** 80 lbs. **3.** 6 (pulleys). **4.** 3 lbs. **5.** 85 lbs.
6. 7 c. 2 qr. 18 lbs. (putting the 6 lbs. pulley highest).
7. 6 pulleys; 64 in. **8.** 4 lbs. **9.** $\frac{1}{5}W$; $7W$. **10.** 10 lbs.: $6\frac{2}{3}$ lbs.
11. The centre of the parallel forces is $\frac{1}{4}$ of a foot from point of suspension of third pulley. **12.** Half his weight.
13. 1 lb. **14.** $\frac{1}{4}P$. **15.** $12\frac{1}{2}$ lbs.
16. 2 tons [neglecting the weight of cable]. **17.** 64 lbs.
18. $280\frac{1}{4}$ lbs. **19.** 15 lbs. **20.** 443 lbs.
22. $8\frac{9}{10}$ times his own weight. **23.** 2635 lbs.; 375 lbs.
25. 56 lbs. **28.** 60 lbs. greatest; 54 lbs. least. **29.** Four.
31. The system in which each pulley is supported by a separate string.

XLIV.

3. $12\sqrt{3}$ lbs. $= 20{\cdot}78$ lbs. **4.** 6 lbs. **5.** 45°. **6.** 30°.
8. As $1 : \sqrt{3}$. **9.** 45°. **10.** 30°. **11.** 60°.
12. 30°; $6\sqrt{3}$ lbs. **14.** 30°; 30°. **15.** As 3 : 4; $2P$.
16. As $AC : AB$. **18.** 1·998... tons = 2 tons very nearly.

XLV.

3. 28 cwt. 1 qr. $17\frac{1}{7}$ lbs. **4.** $186\frac{2}{3}$ lbs. No. **5.** 5 ft.
6. The radius of the wheel which is drawn to the nail makes an angle with the vertical whose sine is $\frac{1}{2}$.
7. (i) when it acts vertically upwards; (ii) when it is vertically downwards.
9. The diagonal to the point of application of the power makes an angle with the vertical whose sine is $\frac{1}{2}\sqrt{2}$.

XLVI.

1. 12 cwt. **2.** 14·85... lbs. **3.** $\frac{11}{10}$ of an inch.
4. $\frac{3}{56} = {\cdot}054...$ of an inch.

XLVII.

1. 5 lbs. **2.** About $28\frac{1}{2}$ lbs. **3.** 20 lbs.
4. About $28\frac{1}{2}$ lbs. **5.** 15 lbs. **6.** 56 lbs.
7. About 2 cwt. 76 lbs. **8.** 50 lbs. **9.** 84 lbs.

XLVIII.

1. $\frac{1}{2}\sqrt{3}$. **2.** 380·8 lbs. **3.** $112 \times \tan \alpha$. **4.** ·577...
5. 60°. **6.** $2 \pm \sqrt{3}$. **7.** $\sqrt{2} \pm {\cdot}1$. **8.** $\frac{1}{2}\sqrt{2}$ lbs.

9. $\tan^{-1}\frac{8}{10}$. **10.** 8 ft. **12.** $\tan^{-1}\frac{1}{2\mu}$; 10 ft.

13. $\frac{1}{240}$ very nearly. **14.** Nearly 5 times.

XLIX.

1. 20 celos. **2.** 80 poundals = weight of $2\frac{1}{2}$ lbs.

3. $\frac{mv^2}{p}$ feet. **6.** The weight of $1026\frac{2}{3}$ lbs.

7. The weight of $22\frac{4}{7}\frac{4}{7}$ lbs.

8. (i) When at rest the tension is 56 lbs. weight.
(ii) When in motion the tension is $(56+70)$ lbs. weight.

9. The difference is about $94\frac{1}{3}$ lbs. weight.

10. At the distance of 3 feet from the 4 lbs.

11. 8 velos. **12.** $\frac{80}{33}$ lbs.

L.

1. $40\times 56\times g$ foot-poundals; 2240 foot-pounds.

2. $10\times 14\times 500\times g$ foot-poundals; 70000 foot-pounds.

3. $10\times 2240\times 500\times g$ foot-poundals; 11200000 foot-pounds.

4. $-12\times 14\times 200\times g$ foot-poundals; -33600 foot-pounds.

5. $300\times 2240\times\frac{7}{80}\times 1760\times 3\times g$ foot-poundals; 70963200 foot-pounds.

6. $224\times 600\times g$ foot-poundals; 134400 foot-pounds.

7. 9 : 880. **8.** 22400 foot-pounds. **9.** $234\frac{4}{9}$ foot-pounds.

10. 50 velos $=34\frac{1}{11}$ miles per hour. **11.** $224\times 8\times 88$ foot-pounds.

12. $19\frac{1}{2}$ miles per hour.

13. $\frac{96}{4+\cos\alpha}$ miles per hour, where $\cos\alpha=\sqrt{\{1-(\frac{1}{80})^2\}}=\cdot 9998\ldots$ that is, $19\frac{1}{2}$ miles per hour, very nearly. **14.** 29568 foot-pounds.

15. 5 : 2.

LI.

1. 80000 foot-poundals. **2.** $(2000)^2\times 4\times 112$ foot-poundals.

3. $(1600)^2\times 6\times 112$ foot-poundals.

4. $(88)^2\times 200\times 2240$ foot-poundals.

5. $\frac{1}{80}\times(88)^2\times 84$ foot-poundals $=813\cdot 12$ foot-pounds.

6. 1680 foot-pounds.

7. $\{(66)^2\times 180\times 2240+3\times 1120\times 1760\times 6\times 32\}$ foot-poundals
$=2240\times 9\times 11\{7920+5120\}$ foot-poundals.

8. $\{(88)^2\times 50\times 2240+500\times 100\times 2240\times 32\}$ foot-poundals
$=2240\times 64\times 50\{121+500\}$ foot-poundals.

9. The necessary work is $112\times 8\times 32$ foot-poundals; the superfluous work is $112\times\frac{1}{2}\times 25$ foot-poundals. Their ratio is 512 : 25.

10. No work is done while the brakes are applied, that is, for 110 yards. The work done in producing kinetic energy is $\frac{1}{2}(11)^2 \times 2240 \times 10$ foot-poundals, the work done against friction is $28 \times 32 \times 1650 \times 3$ foot-poundals. The total work done is

$$2240 \times 11 \{55 + 180\} = 2240 \times 11 \times 235 \text{ foot-poundals.}$$

The work done in going a mile with uniform velocity

$$= 28 \times 32 \times 1760 \times 3 = 2240 \times 11 \times 192 \text{ foot-poundals.}$$

These two are in the ratio 235 : 192.

11. Let the train be m lbs. The work done in the first case is $\{\frac{1}{2}(66)^2 m + \frac{32}{1760} \times m \times 1760 \times 3\}$ foot-poundals, in the second

$$\frac{32}{1760} \times m \times 1760 \times 3 \text{ foot-poundals.}$$

Their ratio is 3586 : 1408 = 163 : 64.

12. $11 \times 2240 \{55 - 12\}$ foot-poundals.

LII.

2. 80 velos.

LIII.

1. $716\frac{2}{3}$ H.P. **2.** 30 miles per hour. **3.** 1·564... **4.** $31\frac{1}{4}$ miles per hour. **5.** $93\frac{3}{4}$ lbs. weight. **6.** $93\frac{3}{4}$ tons. **7.** It does work at the rate of 6·66 H.P. against gravity; it also delivers $\frac{50}{3}$ gallons of water per second with $\frac{50}{3}$ velos; hence it must produce also work at the rate of ($\frac{1}{2}mv^2$, i.e. of) $\frac{1}{2} \times \frac{50}{3} \times (\frac{50}{3})^2 \times \frac{1}{32} \times \frac{1}{550}$ H.P.; that is about 7·2... H. P. **8.** ·1018 H.P.

9. 750 sq. in. [The pressure of the steam is applied to each side of the piston in turn.] **10.** 67·4... lbs. **11.** 7·5. **12.** 9 bus.

LVI.

1. 39·129... inches. **2.** 4 times 39·139... inches. **3.** $\frac{32{\cdot}09}{32{\cdot}19}$ of 39·139... inches. **4.** Let R miles and h miles be the radius of the earth and the height of the mountain respectively, then $(g - g_1) : g = (R)^2 : (R+h)^2$, therefore $1 - g_1/g = (1 + h/R)^{-2} = 1 - 2h/R + \ldots$; hence neglecting squares of h/R; $g_1/g = 2h/R$; and as in Art. 220, $2n/N = g_1/g$, $\therefore\ 2n/N = 2h/R$. Q.E.D. **5.** Here $g - g_1 : g = R - h : R$; whence as in 4 $2n/N = h/R$. **6.** A little more than 7000 feet. **8.** $\frac{11}{12}$ of the seconds pendulum. **9.** Shorten it $\frac{1}{16}$ of an inch. Note; the number of vibrations lost is to the number of vibrations per day as the no. of seconds lost is to the number of seconds per day. **10.** Lengthen it $\frac{1}{12}$ of an inch. **11.** 32·09... + ·22.... **12.** 3600 a/n feet. **13.** He raised the C.G. of the bob and therefore made the clock go faster. **15.** 32·22....

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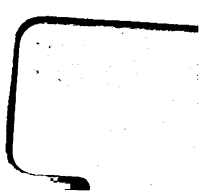

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